Calculus Challenge Exam - June 29, 2021

Department of Mathematics and Statistics
University of New Brunswick

Time: 3 hours Total points: 100

Instructions:

- This exam has 12 questions.
- You have 3 hours to write the exam and 30 minutes to submit your answers into Crowdmark.
- Show your work! Full marks are awarded only when the answer is correct and supported with reasons. Whenever possible, evaluate expressions in your answer (such as $\sin(\pi/2)$, $\ln(e)$, etc).
- Work neatly and in an organized manner.
- Except for the purpose of invigilation, the use of computer, laptop, and other electronic devices is forbidden.
- Notes, books, and other aids of any kind are not permitted.
- Communications with persons other that the instructor or the invigilator are not permitted.
- Answers for each question should be submitted into the designated place on Crowdmark.
- Before pressing the submit button on Crowdmark, make sure that your uploaded pages are in order, rotated correctly, and legible.
- Good luck!

POINTS

1. Find y'. Answers need not be simplified.

   (a) $y = \dfrac{1}{5x^4} + 3^{2x} + 5\sec\left(\dfrac{1}{x}\right) + \cosh(\pi x)$ **(4)**

   (b) $y = \dfrac{\sin^2(x)}{\tan(4x)}$ **(4)**

   (c) $y = \ln\left(xe^x + \dfrac{1}{e}\right)$ **(4)**

   (d) $y = \log_2 x + e^{\sqrt{2-x}}$ **(4)**

   (e) $y = \dfrac{\arctan x}{\sqrt{x^3+1}}$ **(4)**

2. Evaluate the following limits. If the limit is infinite, indicate whether it is ∞ or $-\infty$. If the limit does not exist and is not infinite, write "DNE". You may use L'Hospital's Rule where appropriate. Justify your answers.

   (a) $\displaystyle\lim_{x\to 1} \frac{e^x - 1}{(x-1)^2}$ **(3)**

   (b) $\displaystyle\lim_{t\to 0} \left(\frac{2}{t\sqrt{4+t}} - \frac{1}{t}\right)$ **(3)**

   (c) $\displaystyle\lim_{x\to 0} \frac{x^2 + 2\sin(x)}{5x}$ **(3)**

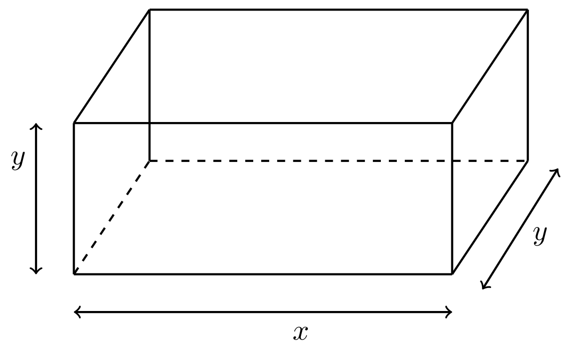
   (d) $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{x^2 + 11x}}{4 - 3x}$ **(3)**

   (e) $\displaystyle\lim_{x\to 0^+} \sqrt{x}\ln x$ **(3)**

3. Use the limit definition of the derivative to find the derivative of $f(x) = \dfrac{1}{x+3}$. Make sure to use proper notation throughout the full solution. **(6)**

4. Consider the curve $(x^2 + y^2)^2 = 25(x^2 - y^2)$. Find $\dfrac{dy}{dx}$. **(5)**

5. A factory is designing rectangular fish tanks with the volume capacity of $800,000$ cm^3 as you see in the diagram below. The material for the sides costs \$2 per cm^2 and the material for the bottom of the tank costs \$6 per cm^2. Find the dimensions of the tank that will minimize material costs. **(7)**
NOTE: Tank has no top.

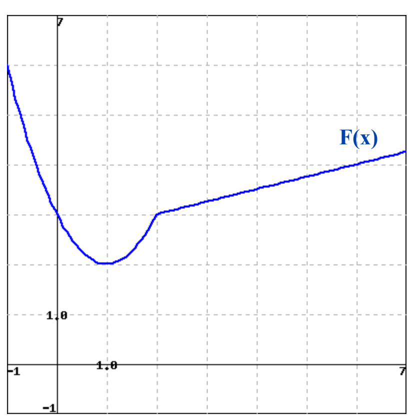


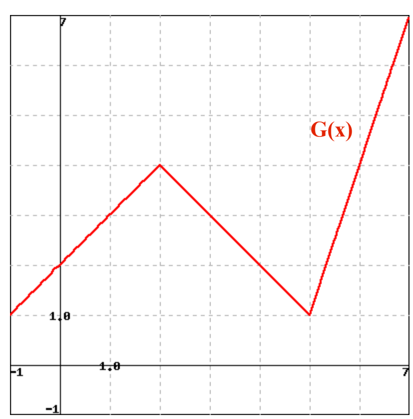
6. A tanker is leaking oil into the ocean. The oil slick spreads and forms a thin film on the water surface. The surface shape of the spill is a circle with radius r, and it has a constant thickness of $h = \dfrac{1}{1000}$ metres (1 millimetre). **(7)**

   (a) Express the volume V of the oil slick as a function of r.

   (b) If the tanker is leaking at the rate of $dV/dt = 100$ cubic metres per hour, how fast is the radius r of the spill increasing when $r = 50$ metres?

7. The graphs of the function F (left, in blue) and G (right, in red) are below. **(6)**
Let $P(x) = F(x)G(x)$, $Q(x) = F(x)/G(x)$ and $R(x) = (F \circ G)(x)$. Calculate the following:





(a) $P'(1)$ (b) $Q'(1)$ (c) $R'(1)$

8. Find the derivative of the following: **(4)**

$$y = 5x^{\cos x}$$

9. Let $f(x) = 2- \mid 2x - 1 \mid$. Show that there is no value of c such that $f(3) - f(0) = f'(c)(3 - 0)$. Explain why this does not contradict the Mean Value Theorem. **(3)**

10. Determine whether the following function is continuous at $x = 0$. Justify your answer. **(6)**

$$g(x) = \begin{cases} e^{1/x} - 2 & \text{if } x < 0 \\ \\ x^2 & \text{if } x \geq 0 \end{cases}$$

11. Find the absolute maximum and the absolute minimum values of $f(x) = 2x - \arcsin(x)$ on $0 \leq x \leq 1$. **(6)**

12. Consider the function $f(x) = \dfrac{4(x+3)^2}{(x-3)^2}$. **(15)**

    (a) Find the domain of f.

    (b) Find any vertical or horizontal asymptotes in the graph.

    (c) Find the intervals on which f is increasing, and the intervals on which it is decreasing.

    (d) Find the coordinates of any local/relative extrema.

    (e) Find the intervals on which the graph is concave up, and the intervals on which it is concave down.

    (f) Find the coordinates of any points of inflection.

    (g) Sketch the graph of $y = f(x)$, taking care to incorporate your answers to (a) through (f) above.

(100)