Student Name (**Print**)

Student Signature

THE UNIVERSITY OF NEW BRUNSWICK

Calculus Challenge Exam

Monday, June 29, 2020 **6:00-9:00pm**

INSTRUCTIONS

1. **Questions begin on Page 2 and continue to Page 13. Check that you have a complete exam.**
2. **The exam is closed book/closed notes. No notes or books are permitted.**
3. **Show all your work and justify your answers. Show all work in the space provided.**
4. **The final exam grade will be converted to a letter grade based on the following conversion scale:**

$A+$	:	[92, 100]	$B+$	:	[75,80)	$C+$	:	[58,65)
A	:	[86,92)	B	:	[70,75)	C	:	[50,58)
$A-$	:	[80,86)	$B-$	:	[65,70)	D	:	[45,50)
						F	:	[0,45)

A letter grade of $B-$ or higher must be achieved on the exam to qualify for credit for Math 1003.

Examiner (**Print**)

Examiner Signature

date

FOR GRADING ONLY

Page	MARK
2	/7
3	/8
4	/5
5	/7
6	/15
7	/10
8	/10
9	/14
10	/9
11-13	/15
TOTAL	/100

Full marks will be awarded only for complete and justified solutions.

[2mks] **1.** If $f(x) = \dfrac{x}{1-x^2}$ and $g(x) = \sqrt{x}$ find $(f \circ g)(x)$ and its domain.

[2mks] **2.** Prove the identity: $\dfrac{2\tan x}{1+\tan^2 x} = \sin(2x)$

[3mks] **3.** Find all values of x that satisfy $e^x - 3e^{-x} = 2$.

4. Find the indicated limits. If the limit is infinite, write $+\infty$ or $-\infty$. If the limit does not exist and is not infinite, write "DNE". You may use L'Hospital's Rule where appropriate. Justify your answers.

[2mks] (a) $\displaystyle \lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

[3mks] (b) $\displaystyle \lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x} \right)$

[3mks] (c) $\displaystyle \lim_{x\to 2} \frac{\ln(2x) - \ln 4}{x^2-4}.$

[2mks] **5.** Show that $f(x) = x - \cos(x)$ has a zero (that is, $f(x) = 0$ has a solution) in the interval $\left[0, \frac{\pi}{2}\right]$. Justify your answer, stating any theorem you used.

[3mks] **6.** Given $f(x) = \begin{cases} \sqrt{1 - \sin x}\ , & x \leq 0 \\ 1 - \dfrac{x}{2} & , \quad x < 0 \end{cases}$, determine whether $f'(x)$ exists for all x.

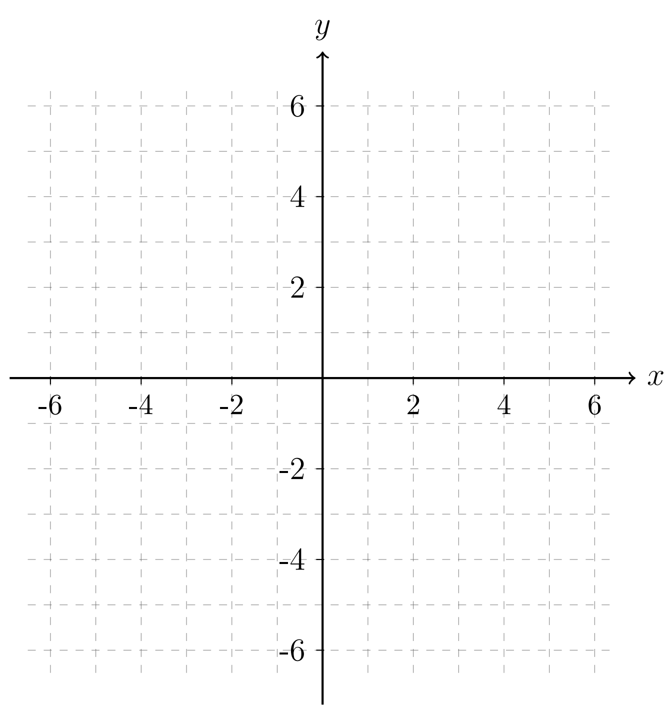
[3mks] **7.** A certain radioactive isotope has a half-life of 500 years. How much of a 100 mg sample will remains after 1500 years? (You do not need to simplify your answer.)

[2mks] **8.** Suppose f is differentiable everywhere, $f'(5) = 0$ and $f''(5) = 2$. Which of the following statements must be true ?

(A) $(5,\ f(5))$ is a local minimum.
(B) $(5,\ f(5))$ is a local maximum.
(C) $(5,\ f(5))$ is the absolute minimum.
(D) $(5,\ f(5))$ is the absolute maximum.
(E) f is decreasing on $(4, 6)$.
(F) none of the above.

[2mks] **9.** Sketch a graph of a function $f(x)$ such that:

- $f(x)$ has a removable discontinuity at $x = -4$,
- $\lim_{x\to 0^-} f(x) = 4$,
- $\lim_{x\to 0^+} f(x) = -4$,
- $f(x)$ is right continuous at $x = 0$, and
- $\lim_{x\to 4^-} f(x) = \infty$.



[15mks] **10.** Find $\dfrac{dy}{dx}$. Do not simplify your answers.

(a) $y = \log_5(x^3) + 3^2$

(b) $y = \sqrt{\cos^{-1}(3x+1)}$

(c) $y = \dfrac{xe^{2x}}{1-x}$

(d) $y = \sin^2(x^2) - 3^{2x}$

(e) $y = \cosh(5x) + 5\sec\left(\dfrac{1}{x}\right)$

[6mks] **11.** (a) If $f(x) + x^2\,[\,f(x)\,]^3 = 10$ and $f(1) = 2$, find $f'(1)$.

(b) Use (a) to find the equation of the line tangent to the curve $y = f(x)$ at the point $(1, 2)$.

[4mks] **12.** If f is a differentiable function and $g(x) = xf(x)$, use the **limit definition** of the derivative to show that $g'(x) = xf'(x) + f(x)$.

[3mks] **13.** Suppose f is continuous on $[2,5]$ and $-3 \leq f'(x) \leq 4$ for all values of x in $(2,5)$. Show that $-9 \leq f(5) - f(2) \leq 12$. State what theorem you use.

[4mks] **14.** Find all horizontal asymptote(s) of $f(x) = \dfrac{2e^x}{e^x - 5}$.

[3mks] **15.** Find the value of $\lim\limits_{x \to \infty} xe^{-x^2}$.

[5mks] **16.** Find the extrema of $f(x) = x - 2\arctan(x)$ on the interval $[0, 4]$.

[4mks] **17.** Use logarithmic differentiation to find $\dfrac{dy}{dx}$ where $y = (3x-1)^{\sqrt{x}}$.

[5mks] **18.** Find the point on the line $y = 2x + 3$ that is closest (has minimal distance) to the origin.

[5mks] **19.** A police car traveling south toward Sussex at 120 km/h pursues a truck traveling east away from Sussex, at 100 km/h (see diagram). At time $t = 0$, the police car is 40 km north and the truck is 60 km east of Sussex. Find the rate at which the distance between the vehicles is changing 15 minutes later.

[4mks] **20.** Solve the initial value problem: $\dfrac{dy}{dx} = 2x - \sin x \quad , \quad y(0) = 5$

[15mks] **21.** Suppose $f(x) = \ln(4 - x^2)$, $f'(x) = \dfrac{-2x}{4 - x^2}$, and $f''(x) = \dfrac{-2(4 + x^2)}{(4 - x^2)^2}$.

This question spans three pages. Find the following information. **Justify your answers by showing your work**. Use the information to sketch the graph of $f(x)$ on the axes provided.

(a) Domain of $f(x)$.

(b) Symmetry of $f(x)$. (Determine whether f is even, odd, or neither.)

(c) Any intercept(s).

(d) Any horizontal asymptotes. Justify your answer.

(e) Any vertical asymptotes. Justify your answer.

(f) Critical numbers.

(g) Open intervals of increase/decrease.

(h) Local extrema (provide points) if any. If there are none, say so.

(i) Open interval(s) where $f(x)$ is concave up/down.

(j) Inflection points, if any. If there are none, say so.

(k) Sketch: