**Last name:** _____ **First name:** _____

**Middle initial(s):** _____

**Date of birth:** _____

**High school:** _____ **Teacher:**_____

Credit at (circle one): **UNB-Fredericton** **UNB-Saint John**

# UNIVERSITY OF NEW BRUNSWICK
# DEPARTMENT OF MATHEMATICS & STATISTICS

*Calculus Challenge Exam*
*Wednesday, June 7, 2017*

Time: 3 hours (no calculators, notes or electronics).

| **Page** | **Points** | **Mark** |
|---|---|---|
| 3 | 20 | |
| 4 | 12 | |
| 5 | 10 | |
| 6 | 10 | |
| 7 | 8 | |
| 8 | 14 | |
| 9 | 7 | |
| 10 | 7 | |
| 11-12 | 12 | |
| **Total:** | 100 | |

## Instructions

1. This exam has 13 pages (including this cover page) and is **printed double-sided**.
2. Do each question in the indicated space. Pages 2 and 13 are intentionally left blank in case you need additional space. If you do, **clearly indicate** where your answers are located. **Please do not separate pages of the exam.**
3. Show your work! Full marks are awarded only when the answer is correct and supported with reasons.
4. Calculators, cell phones, laptops, tablets, and all other electronic devices are **not** permitted. Notes, books, scrap paper and aids of any kind are forbidden.

**EXTRA PAGE – do not remove this page**

**MARKS**

[20] 1. Find $y'$. Answers **need not** be simplified.

(a) $y = 3\left(e^t + \ln t\right)^2$

(b) $y = x^{3/2}\tan(1/x)$

(c) $y = \sec(t^2) + \sin^2 t$

(d) $y = \dfrac{8^x}{x^8} + \log_7(x)$

(e) $y = \arctan\sqrt{1-x}$

**MARKS**

[12] 2. Find the indicated limits. If the limit is infinite, write "$+\infty$" or "$-\infty$". If the limit does not exist and is not infinite, write "DNE". You may use L'Hospital's Rule where appropriate. Justify your answers.

(a) $\lim_{x \to 0^+} \dfrac{\ln(x)}{x}$

(b) $\lim_{x \to -\infty} \dfrac{\sqrt{4x^2 + x + 1}}{x + 1}$

(c) $\lim_{x \to \infty} \dfrac{5x^2}{e^x}$

(d) $\lim_{x \to 3} \dfrac{\sqrt{x^2 - 8}}{x^2 - x - 4}$

**MARKS**

3. Consider the function given by $f(x) = \sqrt{5-x}$.

[7] (a) Use the limit definition of the derivative to compute $f'(1)$ .

[3] (b) Find an equation of the line tangent to the curve $y = \sqrt{5-x}$ at the point $(1, 2)$.

**MARKS**

[5] 4. Consider the curve described by the equation

$$\cos(y^3 - 1) - 5xy = x^3 + 6.$$

Find $\dfrac{dy}{dx}$ (in terms of $x$ and $y$).

[5] 5. Find $\dfrac{dy}{dx}$ when

$$y = (\ln x)^{\ln x}$$

**MARKS**

[8] 6. Consider the function $f$ given by $f(x) = \dfrac{2x^2}{x^2 - 1}$.

(a) The graph of $y = f(x)$ has a horizontal asymptote at $y = 2$. Write down and evaluate a limit that confirms this.

(b) The graph of $y = f(x)$ has a vertical asymptote at $x = -1$. Write down and evaluate a limit that confirms this.

(c) The graph of $y = f(x)$ does not have a vertical asymptote at $x = 0$. Write down and evaluate a limit that confirms this.

(d) The function $f$ has a discontinuity at $x = -1$. Using the definition of continuity, explain why.

**MARKS**

[7] 7. Consider the function $f$ given by $f(x) = 2x^5 + 3x - 1$.

(a) Show that $f(x) = 0$ has at least one (real) solution, that is, there is some number $c$ such that $f(c) = 0$. Justify your answer.

(b) Show that $f(x) = 0$ has exactly one (real) solution, that is, there is only one number $c$ such that $f(c) = 0$. Justify your answer.

[7] 8. A virus spreads through a community according to the model $f(t) = 1000te^{-4t}$ where $f(t)$ is the number of people infected at time $t$, and $t$ is measured in years.

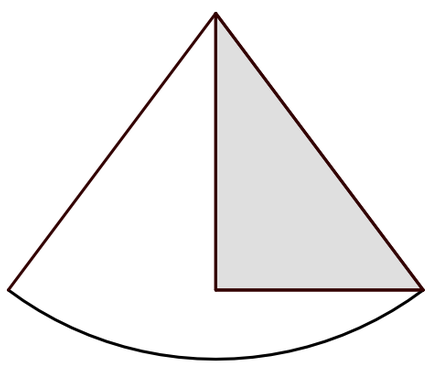
(a) How many people are infected initially?

(b) What happens to the number of people infected in the long term?

(c) Does $f$ have a maximum on the interval $[0, \infty)$? If so, find it. If not, explain how you know this.

**MARKS**

[7] 9. A right triangle with hypotenuse of length 5cm is revolved about one of its legs to create a (right, circular) cone. Maximize the volume of the cone. That is, find the dimensions of the largest possible cone that can be made this way. Note that the volume of a cone with height $h$ and base radius $r$ is give by $V = \frac{1}{3}\pi r^2 h$.



**MARKS**

[7] 10. A rocket is travelling away from Earth. The gravitational force on the rocket, in Newtons, is given by

$$F = \frac{k}{r^2}$$

when the rocket is $r$ km from the centre of the earth. The constant $k$ is $10^{13}$ N $\cdot$ km$^2$. If the force is decreasing at a rate of 5 Newtons per second when the rocket is 10,000 km away from the centre of the earth, how fast is the rocket travelling? (A full solution should include a simplified final answer with units.)

**MARKS**

[12] 11. This question extends over two pages. Answer all parts of this question with regard to the graph of $y = f(x)$, where $f(x) = \dfrac{8x^2}{(x+2)^3}$.

(a) State the domain of $f$.

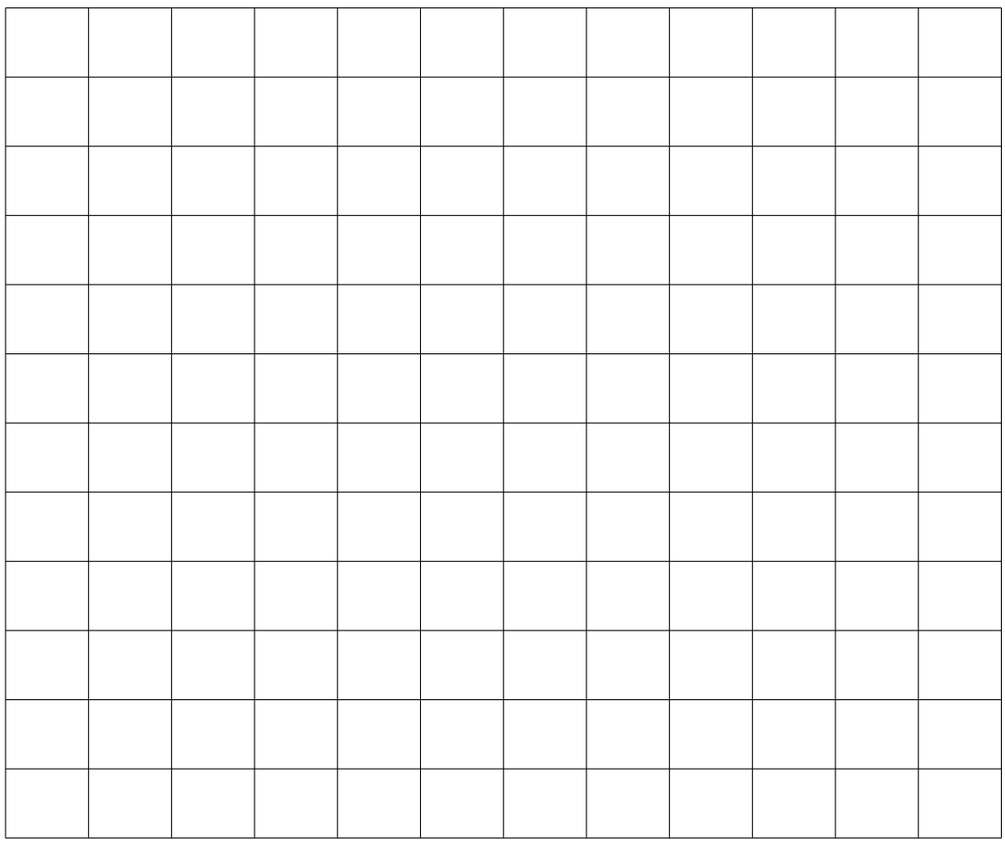
(b) Find the intervals on which $f$ is increasing, and the intervals on which it is decreasing.

(c) Find the coordinates of any local/relative extrema.

**MARKS**

(d) Find the intervals on which the graph is concave up, and the intervals on which it is concave down.

(e) Sketch the graph of $y = f(x)$, taking care to incorporate your answers to (a) through (d) above.



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