ON AN ABSTRACT CLASSIFICATION OF FINITE–DIMENSIONAL HOPF C*-ALGEBRAS

DAN Z. KUČEROVSKÝ

Abstract. We give a complete invariant for finite-dimensional Hopf C*-algebras. Algebras that are equal under the invariant are the same up to a Hopf *-(*co*-anti)isomorphism.

Résumé. Soient A et B deux C*-algèbres de Hopf en dimension fini. Nous définissons une invariante homologique telles que si les algèbres A et B ont les mêmes invariantes, donc A est isomorphe à B ou A est isomorphe à B^{cop}.

1991 *Mathematics Subject Classification.* Primary 47L80, 16T05; Secondary 47L50, 16T20 .

Key words and phrases. Hopf algebras, C*-algebras.

We thank NSERC for financial support.

1. Introduction

It would be desirable to find a classifying functor for Hopf algebras in general. By this we mean an invariant for Hopf algebras that should have good functorial properties and be easy to compute in specific cases. One would also want the invariant to be complete, in the sense that isomorphism at the level of the invariant should induce isomorphism at the level of Hopf algebras. Furthermore, given such an invariant, the self-maps that do not change the value of the invariant should correspond to some interesting class of Hopf algebra automorphisms.

We consider the case of finite-dimensional Hopf C*-algebras. Observing that finite-dimensional and approximately finite dimensional C*-algebras have been classified in the above sense by a type of K-theory[8] suggests that we consider some ring related to Kasparov's ring $\mathrm{KK}_{\widehat{A}}(\mathbb{C},\mathbb{C})$. Instead of working with Fredholm modules, we exploit the finite-dimensionality of our setting and directly use a particular convolution product of finite-dimensional modules. This multiplicative structure, enhanced by some additional information that can also be stated in K-theoretical terms, provides a solution to the above problem (Corollary 4.2) in the case of finite-dimensional Hopf C*-algebras. We should, however, state in advance that we do not obtain anything as strong as Cartan's classification of Lie algebras: in other words, the fact that our classification is abstract means exactly that we do not obtain a list of the finite-dimensional Hopf C*-algebras. Our main results are Theorem 3.1, which generalizes to finite-dimensional Hopf algebras a theorem of Banacheweski's on obtaining a group isomorphism from an antipode-preserving isomorphism of group rings, and a K-theoretical Corollary, Corollary 4.2.

The significance of these isomorphism results lies in the parallel with the C*-algebraic Elliott classification program[8]. Most of the results of the Elliott classification program are obtained from theorems that show that within certain classes of C*-algebra, maps at the level of K-theory or the Cuntz semigroup can be lifted to C*-algebraic maps. It is quite often the case in the C*-algebraic setting that the K-theory groups must be regarded as being augumented by some additional information. The results we obtain for Hopf C*-algebras are of precisely this sort, except that some technically necessary conditions involving co-units and co-centres have been added.

In Section 2 we establish necessary preliminary results, in Section 3 we obtain our first isomorphism result, and in Section 4 we introduce real K-theory, obtaining our second isomorphism result. Our notation is based on that of [1], and we refer to the papers [1, 2] for general background information on Hopf C*-algebras. In general, we denote co-products by Δ, antipodes by κ, and Haar states by τ. In our setting the Haar states are tracial. By the term *cotracial linear functional*, we mean a linear functional g having the property that $g(a \diamond b) = g(b \diamond a)$, where $\diamond$ denotes convolution. Similarly, a *tracial linear functional* is a linear functional T having the property that $T(ab) = T(ba)$. Let us say that a linear space isomorphism is a *bi-algebra isomorphism* if it intertwines both the algebra products and the co-products. It must then map unit to unit and co-unit to co-unit. Let us say that a linear space isomorphism is a *bi-algebra* co-*anti-isomorphism* if it intertwines the algebra products and flips the co-product. Such a map will preserve units and co-units. The terminology is motivated by the fact that a bi-algebra *co*-anti-isomorphism is equivalent to a bi-algebra isomorphism where one of the bi-algebras has been replaced by its co-opposite bi-algebra. (A co-*anti-automorphism* is a linear space automorphism preserving the product and flipping the co-product. If it also preserves the C*-algebraic involution, it is then a *-co-*anti-automorphism.*) When working with nonzero linear maps that are bi-algebra maps of Hopf algebras, it is frequently convenient to use the fact that such maps intertwine antipodes[7, page 152]. This can be summarized by saying that bi-algebra maps of Hopf algebras are Hopf algebra maps. A Hopf algebra is said to be *commutative* if the multiplicative structure is abelian, and to be *co-commutative* if the co-product is invariant under the tensor product flip. The *co-centre* of a Hopf algebra is the sub-algebra of elements having the property that their image with respect to the co-product homomorphism is invariant under the flip. Group algebras and their duals provide examples of Hopf algebras; however, there certainly exist finite-dimensional Hopf algebras that are not related to group algebras. The eight-dimensional Kats-Pal'yutkin example[14]—see also the discussion in [2, page 474]—is in this class.

2. A fusion product ring

In the finite-dimensional case, the K-theory group $\mathrm{K}(A)$ is generated by projections that are minimal in the algebra A. Projections p_i give projective modules over A of the form p_iA. A convolution product of two such projections, in terms of modules, can be defined in terms of pullback by the co-product homomorphism:

$$\Delta_*(p_1A \otimes p_2A),$$

where the notation Δ_* indicates restriction of rings with respect to the co-product homomorphism. The resulting A-module is in fact projective because the algebra A is semi-simple, and thus defines a class in the K-theory group $\mathrm{K}(A)$. At the level of K-theory, a general projection defines a class in $\mathrm{K}(A)$ which can be written in terms of a sum of the one-dimensional generators of $\mathrm{K}(A)$, and the product of two such projections can be defined in terms of the above product operation on the generators—the functoriality of the algebraic restriction of rings operation ensures that the product is independent of the choice of generators. Let us introduce the notation $\square$ for this product operation.

It is also possible to view this product in terms of operators rather than modules. In the finite-dimensional case, the product takes a pair of elements belonging to A to an element of $A \otimes M_n$. In order to accommodate different sizes of matrix algebra in a unified way, we may as well regard M_n as a subalgebra of the compact operators $\mathcal{K}$, and then make use of the fact that $\mathcal{K} \otimes \mathcal{K}$ is isomorphic to $\mathcal{K}$. In this way, we obtain a ring whose elements are matrices over A, with a product operation that is determined in a natural way by the co-product homomorphism. At the level of operators, the product has the properties

(i) (Pullback) $(a \otimes \mathrm{Id})(b \,\square\, c) = (ba_1) \,\square\, (ca_2)$ if $\Delta(a) = a_1 \otimes a_2$, and
(ii) (normalization) $(\tau \otimes \tau)(b \,\square\, c) = \tau(b)\tau(c)$, where τ denotes the extension of the Haar state of A to the enveloping $B(H)$, where H is a finite-dimensional Hilbert space.

We may consider K-theory maps $\phi : \mathrm{K}(A) \to \mathrm{K}(B)$ that intertwine the product $\square_A$ and $\square_B$. It is convenient to state the definition of such maps in a dual form, in terms of traces on K-theory:

Definition 2.1. *A map $f \colon \mathrm{K}(A) \to \mathrm{K}(B)$ is said to be* K-co-multiplicative *if $(\sigma \otimes \tau)[(f \otimes \mathrm{Id})(p \,\square\, q)] = (\sigma \otimes \tau)[f(p) \,\square\, f(q)]$, where p and q are minimal projections of A, the linear functional σ is a trace on B, and the linear functional τ is the standard trace on $\mathcal{K}$.*

It is sufficient to consider only operators on finite-dimensional Hilbert spaces in the above definition.

We note that the Hopf algebra antipode κ is an anti-homorphism, and thus induces a map at the level of K-theory. In fact, since the antipode has the property $\Delta \circ \kappa = \sigma \circ (\kappa \otimes \kappa) \circ \Delta$, where σ is the flip, it follows that $\kappa\,(p \,\square\, q) = \kappa(q) \,\square\, \kappa(p)$. Therefore, at the level of K-theory, the antipode reverses the product $\square$, and thus is an example of a K-co-multiplicative map from the K-theory of A to the K-theory of the co-opposite bi-algebra A^{cop}.

Finally, we define an operator-valued Fourier transform, $\mathcal{F} \colon A \to \widehat{A}$, by the equation $\beta(a, \mathcal{F}(b)) = \tau(ab)$, where a and b are elements of a

Hopf C*-algebra A, the pairing with the dual algebra $\widehat{A}$ is denoted β, and τ is the Haar state.

Lemma 2.2. *Let A, B be unital Hopf C*-algebras with tracial Haar states τ_A and τ_B. Let $f : A \to B$ be a Jordan *-monomorphism that intertwines τ_A and τ_B. Then*

$$f^* \circ \mathcal{F}_B \circ f = \mathcal{F}_A.$$

Proof. The Fourier transform on A can be written as $\mathcal{F}\colon a \mapsto \tau_A(a\,\cdot)$. A similar statement holds for the Fourier transform on B. Then, using the property that $\tau_B(f(x)) = \tau_A(x)$ for all $x \in A$, we get

$$\begin{aligned} f^* \circ (\tau_B(f(a)\ \cdot)) &= \tau_B(f(a)f(\,\cdot\,)) \\ &= \tau_B(f(a(\,\cdot\,))) \\ &= \tau_A(a\ \cdot). \end{aligned}$$

It follows that $f^* \circ \mathcal{F}_B \circ f = \mathcal{F}_A$. □

We use the above Lemma to prove the following:

Lemma 2.3. *Let A and B be finite-dimensional Hopf C*-algebras. Let $f : A \to B$ be a C*-isomorphism. If the map induced on* K*-theory by f respects the product $\square$, then the map f^* induced by f on the dual algebra satisfies $g(f^{*-1}(y_1y_2)) = g(f^{*-1}(y_1)f^{*-1}(y_2))$ for all $y_i \in \widehat{A}$ and all cotracial linear functionals g.*

Proof. We are given that $\bar{f} = (f \otimes \mathrm{Id})\colon A \otimes \mathcal{K} \to B \otimes \mathcal{K}$ respects $\square$, at the level of K-theory. Thus, in terms of formulas we have:

$$(\sigma \otimes \tau)[\bar{f}(p \,\square\, q)] = (\sigma \otimes \tau)[f(p) \,\square\, f(q)],$$

where p and q are minimal projections of A, the linear functional σ is a trace on B and the linear functional τ is the standard trace on $\mathcal{K}$, which is in fact an extension of the Haar state to the GNS representation of B. Let us define an binary operation $\diamond$ by $\mathcal{F}(p \diamond q) = \mathcal{F}(p)\mathcal{F}(q)$. Since $\tau(p \diamond q) = \tau(p)\tau(q)$ and $(\tau \otimes \tau)(p \,\square\, q) = \tau(p)\tau(q)$, we have that for minimal projections p and q in A, $(\tau \otimes \tau)(p \,\square\, q) = \tau(p \diamond q)$. Considering a trace σ of the form $\tau(z\cdot)$ where z is a central projection of A, the pullback property of the product operation gives $(\sigma \otimes \tau)(p \,\square\, q) = \sigma(p \diamond q)$. We deduce from the fact that f respects the product $\square$ that therefore $f(p) \diamond f(q)$ and $f(p \diamond q)$ have the same value under the traces of B.

Passing to linear combinations of traces, we thus have, for all tracial linear functionals T on B, that

$$T(f(p) \diamond f(q)) = T(f(p \diamond q)).$$

Now, we note that since f is a C*-algebraic isomorphism, and since the Haar states are determined by the block structure of the C*-algebras A and B, it follows that f intertwines the Haar states of A and B. We may then use Lemma 2.2 to express the action of f on the dual algebras in terms of the Fourier transforms, obtaining $f = \mathcal{F}^{-1} \circ f^{*-1} \circ \mathcal{F}$. Inserting this into the above, we conclude that, on denoting the Fourier transforms of p and q by $\hat{p}$ and $\hat{q}$ respectively,

$$g(f^{*-1}(\hat{p})f^{*-1}(\hat{q})) = g(f^{*-1}(\hat{p}\hat{q}))$$

for any cotracial functional g.

Regarding the above as an equality of bi-linear forms, we note that in a finite-dimensional C*-algebra the minimal projections generate the algebra linearly. The Fourier transform will in the finite-dimensional case take such a subset to a similar subset, and we conclude that

$$g(f^{*-1}(y_1 y_2)) = g(f^{*-1}(y_1)f^{*-1}(y_2)),$$

for any cotracial functional g and any $y_i \in \widehat{A}$. □

Proposition 2.4. *Let A and B be finite-dimensional Hopf C*-algebras. Let $f\colon A \to B$ be a C*-algebraic isomorphism that intertwines co-units, co-centres, and antipodes. Suppose also that the induced map on* K*-theory is* K*-co-multiplicative,* i.e. *the induced map respects the product* $\square$. *Then the pullback map f^* is a Jordan *-homomorphism.*

Proof. Applying Lemma 2.3 to f^{-1}, we have that f^* is at least multiplicative under cotracial linear functionals. Furthermore,

$$g(f^*(b_1)f^*(b_2)f^*(b_3)\cdots f^*(b_n)) = g(f^*(b_1 b_2 b_3 \cdots b_n)) \tag{1}$$

for all cotracial functionals g. We note that f^* is a linear and unital map that maps centre to centre, co-centre to co-centre, self-adjoint elements to self-adjoint elements, intertwines antipodes, and intertwines Haar states. We deduce from the above equation that

$$\tau(f^*(z)^n) = \tau(f^*(z^n)),$$

where z is any element of the centre of $\widehat{B}$, and τ is the Haar state (which has the property that $\tau(a \diamond b) = \tau(a)\tau(b)$ and thus is a cotracial linear functional.)

If we restrict f^* to a map from the centre of $\widehat{B}$ to the centre of $\widehat{A}$, we then have a map of finite-dimensional abelian C*-algebras. But since this map preserves not just the trace, but the trace of every power, since $\tau(f^*(z)^n) = \tau(f^*(z^n))$, this forces the map f^* to preserve the range of z. In other words, z is a complex-valued function on a finite set, and $f^*(z)$ has the same set of values up to a permutation. Since this holds true for every z in the centre, it follows[9] that the unital map f^* is a homomorphism when restricted to the centre.

Next, we consider the behaviour of f^* when restricted to a matrix block of $\widehat{B}$. The unit of this matrix block is an idempotent z of the centre, and $z' := f^*(z)$ is the unit of some matrix block of $\widehat{A}$. It is convenient to consider the map ϕ of matrix blocks given by $b \mapsto z' f^*(b)$. Equation (1) and the fact that z' is a central idempotent implies that $\tau(\phi(b)^n) = \tau(b^n)$. With b taken to be a Hermitian matrix, the element $\phi(b)$ is a Hermitian matrix. These matrices have the property that when raised to a power, the corresponding powers are equal under the trace. By the classical Newton-Girard formulas, this is sufficient to insure that the characteristic polynomials of these matrices have their corresponding elementary symmetric functions equal, and this in turn implies that the eigenvalues of these matrices are the same up to permutation. Now, we recall the result of Marcus and Moyls[16, Theorems 3 and 4], that unital linear maps of matrix algebras which take Hermitian matrices to Hermitian matrices and which preserve the spectrum of Hermitian matrices are in fact of the form $b \mapsto U^* b U$ or of the form $b \mapsto U^* b^T U$, where U is a unitary and b^T denotes the ordinary transpose of the matrix b. In either of these cases, we conclude that ϕ is in fact a Jordan *-homomorphism. We thus have that f^* is a direct sum of Jordan *-homomorphisms on matrix blocks, and therefore is a Jordan *-homomorphism. □

Lemma 2.5. *Let A and B be finite-dimensional Hopf C*-algebras with tracial Haar states. Let $\alpha \colon A \to B$ be a *-isomorphism, and denote by $\hat{\alpha} \colon \widehat{B} \to \widehat{A}$ its induced action on the duals. Suppose that the action $\hat{\alpha}$ on the duals is a Jordan *-isomorphism. Then either $\hat{\alpha}$ is multiplicative, or $\hat{\alpha}$ is anti-multiplicative.*

Proof. A Jordan *-isomorphism maps the C*-algebraic unit ball onto the unit ball. If $\hat{\alpha}$ maps the unit ball onto the unit ball, this implies that the map that it induces on linear functionals is an isometry with respect to the usual dual norm on linear functionals. Thus, the map α is an isometry with respect to this norm, and thus is an isometry between pre-duals in the sense of [4, Théorème 2.9]. It follows by [4, Théorème 2.9] that $\hat{\alpha}$ is either multiplicative or anti-multiplicative, as asserted. □

3. A generalization of Banacheweski's theorem

Theorem 3.1. *Let A and B be finite-dimensional Hopf C*-algebras. Let $f \colon A \to B$ be a co-centre preserving C*-algebraic isomorphism that* *intertwines antipodes and co-units. We suppose that the induced map on* K*-theory intertwines the* K*-theory products* $\square_A$ *and* $\square_B$*. Then* A *and* B *are isomorphic or* co-*anti-isomorphic as Hopf algebras.*

Proof. By Proposition 2.4 we have that under the hypothesis the pullback $f^*\colon \widehat{B} \to \widehat{A}$ of f is a Jordan *-isomorphism at the level of C*-algebras. By Lemma 2.5 the pullback map f^* is either multiplicative or anti-multiplicative. We thus have, by duality, that f is either an isomorphism or a *co*-anti-isomorphism of bi-algebras. It is shown in [7, page 152] that a bi-algebra map of Hopf algebras intertwines the Hopf algebra antipodes, and thus is a Hopf algebra map. It therefore follows that if f is a bi-algebra isomorphism, it is also a Hopf algebra isomorphism. On the other hand, in the case that the pullback map f^* is anti-multiplicative, the map f is a bi-algebra isomorphism of A with the co-opposite bi-algebra to B, and we can again conclude that f intertwines antipodes. It follows in either case that f is a Hopf algebra (*co*-anti)isomorphism (*i.e.,* one or the other). □

4. Real K-theory

Let us first recall some facts about real K-theory and real C*-algebras. Denote by K_R the Banach algebra K-theory groups in the real case. In the complex finite-dimensional case there is only one nonzero K-group up to periodicity (see, for example, the discussion in [3, Chapter 4]). In the case of a finite-dimensional algebra over the quaternions, there are four nonzero groups after periodicity[10, Theorem 4.6], and in the purely real case there again are four nonzero K-groups[ibid.]. However, not all of the real K-groups need to be explicitly considered in order to obtain a classification by K-theory. It follows from [10, Theorem 8.3] that if there are given group isomorphisms of the K_R groups $\mathrm{K}_{R,0}, \mathrm{K}_{R,2}$, and $\mathrm{K}_{R,4}$, one can obtain a C*-algebraic isomorphism of finite-dimensional real C*-algebras.

We now recall that in our setting the antipode is determined by (and determines) a certain real subalgebra. The next Lemma collects some known facts.

Lemma 4.1. *In a finite-dimensional Hopf C*-algebra* A *the antipode* κ *is involutive and is determined by the fixed point real subalgebra* $R := \{x | \kappa^*(x) = x\}$*. We have* $R + iR = A$*,* $R = R^* = \kappa(R)$*, and* $R \cap iR = \{0\}$*. The co-centre* $\mathrm{CoZ}(A)$ *is spanned by* $\mathrm{CoZ}(A) \cap R$*.*

Proof. In the finite dimensional case, the antipode is an involutive antiautomorphism, and $\kappa(x^*) = \kappa(x)^*$. This is shown in, for example, [1, paragraph 1.2]; see also [2, page 448]. Hence R is a real algebra, and $R^* = R$. It follows that $R = \kappa(R)$.

In the linear functional picture of A, the subalgebra R is given by self-adjoint linear functionals (this description of R follows immediately from [1, paragraph 1.2]). Since any linear functional f can be written as a linear combination $if_I + f_R$ of self-adjoint linear functionals f_I and f_R we have $R + iR = A$ as claimed, and the set $R \cap iR$ is given by elements whose associated linear functionals are both self-adjoint and skew-adjoint. But the only such linear functional is the zero functional. The claim that the co-centre $\mathrm{CoZ}(A)$ is spanned as a complex vector space by $\mathrm{CoZ}(A) \cap R$ follows from the description of the co-centre given in [1, Proposition 1.5].

The antipode may now be defined in terms of the algebra R by defining an involutory antilinear map $J\colon A \to A$ by $J(r + is) = r - is$, where r and s are in R. We then have $J(x) = \kappa(x)^*$, and this equation lets us recover the antipode from the algebra R. □

It follows from Lemma 4.1 that a C*-algebraic isomorphism of finite-dimensional Hopf C*-algebras intertwines antipodes if and only if it maps the κ-symmetric elements onto the κ-symmetric elements. Since K_R-theory is a classifying functor for approximately finite real C*-algebras[10, 19, 12, 8], once suitable definitions are made, maps of the K_R groups determine algebra *-homomorphisms, and every C*-algebraic *-homomorphism determines a K_R-group map. In particular, we can use isomorphisms of the real K-groups of the κ-symmetric elements to define C*-algebraic *-isomorphisms that intertwine the Hopf algebra antipodes. Finally, we note that a map of real K-theory groups induces a map of the usual complex K-groups, and that the Haar state and the co-unit are completely determined by the maps that they induce on the real (or the complex) K-theory groups. The induced map is generally called a K-theory state.

In the next Corollary, the notation $\mathrm{CoZ}_{\mathbb{R}}$ denotes the real sub-C*-algebra of κ-symmetric co-commutative elements.

Corollary 4.2. *Finite dimensional Hopf C*-algebras are classified up to Hopf algebra (*co*-anti-)isomorphism by the invariant*

$$\mathrm{K}_R(\mathrm{CoZ}_{\mathbb{R}}(A)) \longrightarrow \mathrm{K}_R(R_A) \longrightarrow \mathrm{K}(A).$$

The maps between the invariants induce an intertwining of the products on the $\mathrm{K}(A)$ *groups, and intertwine the* K*-theory state coming from the co-unit* ϵ.

Proof. Given an intertwining at the K-theory level, of the form

$$\begin{array}{ccccc} \mathrm{K}_R(\mathrm{CoZ}_{\mathbb{R}}(A_1)) & \longrightarrow & \mathrm{K}_R(R_{A_1}) & \longrightarrow & \mathrm{K}(A_1) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{K}_R(\mathrm{CoZ}_{\mathbb{R}}(A_2)) & \longrightarrow & \mathrm{K}_R(R_{A_2}) & \longrightarrow & \mathrm{K}(A_2) \end{array},$$

the classification theory discussed above gives us a C*-algebraic isomorphism $f\colon A_1 \to A_2$ that maps R_{A_1} onto R_{A_2}, and maps $\mathrm{CoZ}_{\mathbb{R}}(A_1)$ onto $\mathrm{CoZ}_{\mathbb{R}}(A_2)$. It then follows from Lemma 4.1 that f intertwines antipodes and maps co-centre onto co-centre. In addition, f necessarily intertwines co-units when restricted to projections, and thus intertwines co-units in general. Theorem 3.1 then provides a Hopf algebra (*co*-anti)isomorphism of the Hopf algebras A_1 and A_2.

□

We observe that the above classifying map inherits the functorial properties of the K-theory groups.

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University of New Brunswick
Canada E3B 5A3
E-mail address: dkucerov@unb.ca