ON THE UNIVERSALITY OF THE von MISES TRANSFORMATION

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Abstract: The concept of a universal approach to coordinate transformation is analyzed in this work in an attempt to show that the resulting system of transformed equations is under-determined. Solutions obtained using this concept do not necessarily satisfy the equation of continuity. A condition under which the continuity equation is satisfied is stated in this work.

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1 Introduction

It can be argued that fluid flow in Cartesian domains is the exception rather than the rule. Realistic fluid flow situations involve flow domains possessing complex geometries. Flow over wings, flow through natural media (groundwater flow), flow through curvilinear channels and in rivers, and flow through aircraft intake systems, are examples of natural and man-made complex geometries in which one is required to analyze the dynamics of fluid flow.

The irregularity of the boundaries of the above flow domains, the difficulty at times of describing the boundaries mathematically, accompanied with the absence of a general coordinate system capable of describing all complex geometries hinder the detailed analysis of fluid flow and can lead to various approximations that result in inaccuracies in the solutions obtained. Even in the case of two-dimensional geometric domains the presence of curvilinear boundary adds to the challenge of solving the already troublesome boundary value problems involving the essentially non-linear governing equations.

In the case of two-dimensional fluid flow in curved domains, the absence of closed form solutions, the availability of geometrical mapping techniques accompanied with the advent of high speed computers have given computational fluid dynamics a status as a viable alternative to analytical methods and a means of providing analysis to complex fluid flow problems. An enormous amount of literature has been devoted to this subject matter. Of interest to the current work are the techniques of generating Cartesian computational domains in order to facilitate the use of finite differences. A special class of these techniques are the following methods:

1) The curvilinear coordinate transformation $(x, y) \to (\xi, \eta)$.
   The coordinates ξ and η may or may not have a relationship with the physical quantities being solved for. Their relationship to the physical boundaries is typically through the computational grid that is generated either numerically or algebraically, with the physical boundaries representing conditions on the grid generator. Grid generation has been extensively discussed by various authors (see [6] and the references therein).

2) The $\varphi - \psi$ transformation $(x, y) \to (\varphi, \psi)$.
   The coordinate ψ represents the streamfunction of the two-dimensional flow, while φ remains an arbitrary function to be solved for. In (φ, ψ) nets, however, φ is the velocity potential. Details of the use of the $\varphi - \psi$ transforma-

tion in the study of viscous fluid flow can be found in the work of Martin (see [9]).

3) The unidirectional coordinate transformation $(x, y) \rightarrow (x, \eta)$
In the study of some fluid flow problems (for example, flow over an airfoil; flow through channels with curved upper and/or lower boundaries), it is desirable to maintain the physical length of the channel or keep the location of the leading and trailing edges of an airfoil known and unchanged. The physical variable x is thus kept unchanged while the lateral curvilinear coordinate line is transformed into a straight line $\eta =$constant. This approach proved to be useful in the study of some viscous fluid flow problems (see [8] and the references therein).

4) The von Mises transformation $(x, y) \rightarrow (x, \psi)$.
While the von Mises transformation has been in existence for close to eight decades, its implementation in computational fluid dynamics came about only two decades ago with the pioneering work of Barron (see [2]). In this transformation, the physical curvilinear flow domain in the (x, y) plane is mapped onto the rectangular (x, ψ) computational domain, where the lines $\psi =$constant represent the streamlines of the flow.

Given a fluid flow problem in a curvilinear domain, the domain is first mapped; the governing equations and boundary conditions are transformed into the new variables; a solution is then obtained in the computational domain for the variable $y(x, \psi)$; the streamlines are then plotted by plotting the computed y values for a given value of ψ. This approach has received considerable attention in the study of inviscid and viscous fluid flow, and milestones have been achieved in the study of single- and multi-phase flow (see [1, 3, 5]).

Of particular interest to the current work is the concept of universality of the von Mises transformation, which was coined by Hamdan and Ford [7]. The claim is that the transformation is capable of generating a viscous fluid flow solution that is valid for all two dimensional geometries. Their approach has been one where the equations governing viscous fluid flow are first cast in velocity-vorticity form followed by implementation of the von Mises transformation. The von Mises transformation results in three coupled partial differential equations to be solved (two velocity and one vorticity equation) instead of four scalar equations. The authors concluded that three of the equations are independent of the physical plane geometry, and thus they can be solved once. The fourth equation (an equation for $y(x,\psi)$) is therefore the only equation that needs to be solved, and is the equation that produces different streamline patterns for flows in different geometries.

While the work [7] is optimistic, and in fact the concept of universality is rather attractive, there are modifications to the approach and to the solution algorithm that need to be addressed. At the outset, concern arises as to whether the equation of continuity is in fact satisfied when the universal approach is utilized. These issues are addressed in this work.

2 Viscous Fluid Flow Equations

Consider the flow in a two-dimensional, dimensionless channel bounded below and above by solid, curvilinear boundaries. The channel is described by: $\{(x,y) \,|\, f_1(x) \le y \le f_2(x); a \le x \le b\}$, where $f_1(x)$ and $f_2(x)$ are known smooth functions.

The flow of a viscous, incompressible fluid through the channel is governed by the equations of continuity and linear momentum, which take the following dimensionless form, respectively:

$$\nabla \cdot \mathbf{v} = 0 \tag{1}$$

$$(\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{Re}\nabla^2 \mathbf{v} \tag{2}$$

wherein, for two-dimensional flow, $\mathbf{v} = (u, v)$ is the velocity vector field, p is the pressure, ρ is the fluid density, Re is the Reynolds number,$\nabla \equiv (\partial_x, \partial_y)$,$\nabla^2 \equiv \partial_{xx} + \partial_{yy}$ and subscript notation denotes partial differentiation.

Equations (1) and (2) represent a system of three scalar equations that are to be solved for the primitive variables $u(x,y)$, $v(x,y)$, and $p(x,y)$, subject to the no-slip velocity boundary conditions on the solid lower and upper boundaries. At the inlet and exit of the channel, the normal velocity component vanishes, while the tangential velocity component can be taken as a function of y (assuming the channel is long enough to allow for this condition to be imposed at the exit). Appropriate Neumann conditions are usually imposed on the pressure.

The continuity equation (1) implies the existence of a streamfunction $\psi(x,y)$ such that:

$$\psi_y = u \tag{3}$$

and

$$\psi_x = -v \tag{4}$$

Elimination of the pressure term from equation (2) and the introduction of vorticity, $\omega(x, y)$, that is defined in terms of the velocity components as

$$\omega = v_x - u_y \tag{5}$$

facilitates casting the flow equations in the following vorticity-streamfunction form:

Streamfunction equation:

$$\psi_{xx} + \psi_{yy} = -\omega \tag{6}$$

Vorticity equation:

$$\omega_{xx} + \omega_{yy} = Re\{\psi_y\omega_x - \psi_x\omega_y\}. \tag{7}$$

In vorticity-streamfunction, the problem has been reduced from that of solving three equations in three primitive variables to solving two equations in the two unknowns ψ and ω . Furthermore, elimination of the pressure from the momentum equation eliminates the problem of handling pressure boundary conditions and, in case of flow through a two-dimensional channel, exit pressure conditions.

We remark here that the introduction of the streamfunction results in the continuity equation (1) being automatically satisfied. The streamfunction enjoys Dirichlet conditions when the solid boundaries are taken as streamlines of the flow, say $\psi = \psi_{\min}$ on the lower boundary and $\psi = \psi_{\max}$ on the upper boundary. At the inlet and exit of the channel, the Dirichlet conditions on ψ are determined from the inlet and exit tangential velocity profile, by integrating equation (3).

The absence of explicit vorticity boundary conditions necessitates imposing appropriate inlet, exit and solid boundary conditions. Assuming that equation (5) is valid on all boundaries then, with $v_x = 0, \omega = -u_y$ can be used on all boundaries of the configuration at hand (even at the exit of the channel, since the profile of u is assumed to be known there). Alternatively, w_x can be taken to be zero at the exit of the channel.

Extension of the vorticity-streamfunction formulation to three dimensional flows requires the introduction of a stream-surface or the use of two streamfunctions. A viable alternative therefore is to cast the equations in velocity-vorticity form. The two-dimensional version corresponding to the governing equations at hand is as follows:

Vorticity equation:

$$\omega_{xx} + \omega_{yy} = Re\{u\omega_x + u\omega_y\}. \tag{8}$$

Velocity equations:

$$u_{xx} + u_{yy} = -\omega_y \tag{9}$$

$$v_{xx} + v_{yy} = \omega_x. \tag{10}$$

In this formulation, it is required to solve the three equations, (8-10), for the three unknowns (two velocity components and the vorticity).

We remark here that solution to equations (8-10) does not automatically satisfy the equation of continuity. This problem has been circumvented by imposing the continuity equation (1) on the boundaries (cf.[4]and the references therein). However, the trade-off is the ease of extension of the formulation to three-dimensional flow and the absence of the pressure from the equations.

3 The von Mises Transformation

Consider the curvilinear net (x, ψ) in which the curves ψ = constant represent the streamlines of the flow. Now consider the transformation $(x, y) \to (x, \psi)$, defined by

$$y = y(x, \psi). \tag{11}$$

The Jacobian of transformation is given by

$$J = \left|\frac{\partial(x, y)}{\partial(x, \psi)}\right| = y_\psi \tag{12}$$

If $0 < J < \omega$, then the inverse transformation exists and the following first partial derivative operators in the two coordinate systems are obtained:

$$\partial_x = \partial_x - \frac{y_x}{y_\psi}\partial_\psi \tag{13}$$

$$\partial_y = \frac{1}{y_\psi}\partial_\psi \tag{14}$$

Second partial derivative operators can be obtained by applying operators (13) and (14) onto themselves.

4 Transforming the Vorticity-Streamfunction Equations

Applying the above transformation to the vorticity-streamfunction equations (6) and (7), respectively, yields:

$$L_1(y) = \omega(y_\psi)^3 \tag{15}$$

$$L_1(\omega) = \omega\omega_\psi(y_\psi)^2 + Re\, y_\psi \omega_x \tag{16}$$

where

$$L_1 \equiv (y_\psi)^2\partial_{xx} - 2y_x y_\psi \partial_{x\psi} + [1 + (y_x)^2]\partial_{\psi\psi}. \tag{17}$$

The velocity components defined by equations (3) and (4) take the following forms, respectively, in the new coordinate system:

$$u = \frac{1}{y_\psi} = \frac{1}{J} \tag{18}$$

$$v = \frac{y_x}{y_\psi} = uy_x = \frac{y_x}{J}. \tag{19}$$

We remark here that the continuity equation (1) which is automatically satisfied in the vorticity-streamfunction $\omega - \psi$ formulation is also automatically satisfied in the von Mises $\omega - y$ formulation. This can be seen as follows.

The continuity equation (1) takes the following form in the von Mises coordinates, obtained by applying operators (13) and (14) to (1):

$$u_x + uv_\psi - vu_\psi = 0. \tag{20}$$

Since the inverse of the transformation exists ($0 < J < \omega$ in the flow field, and J is of the same sign when there is no flow separation or flow reversal) then $y(x,\psi)$ satisfies the integrability condition $y_{x\psi} = y_{\psi x}$. It follows that $u(x,\psi)$and $v(x,\psi)$, defined in (18) and (19), satisfy (20). We therefore make the following observation.

Observation 1:

If $u(x,\psi)$ and $v(x,\psi)$ are as defined by equations (18) and (19) then the equation of continuity (20) is satisfied.

We note that on a solid boundary, the no-slip condition forces J to be infinite. This implies that if the continuity equation (20) is to be satisfied on the solid boundary, then the vorticity must be redefined. Assuming that equation (15) is valid in the domain and on the boundary, then upon using equations (18) and (19) in (15), we obtain:

$$\omega = v_x + \left(\frac{v^2}{u} - u\right) u_\psi - \frac{v}{u} u_x - 2vv_\psi. \tag{21}$$

If the continuity equation (20) is to be satisfied, then an expression for u_x obtained from (20) is used in (21) to yield:

$$\omega = v_x - \frac{1}{2}(q^2)_\psi \tag{22}$$

where

$$q^2(x,\psi) = u^2(x,\psi) + v^2(x,\psi). \tag{23}$$

On a solid boundary,$v_x = 0$ and equation (22) reduces to:

$$\omega = -\frac{1}{2}(q^2)_\psi. \tag{24}$$

The above analysis establishes the following observation.

Observation 2:

In the von Mises coordinates, if the continuity equation (20) is satisfied then the boundary vorticity is given by equation (24).

With this transformation, the problem of solving equations (6) and (7) for $\psi(x, y)$ and $\omega(x, y)$ in the physical curvilinear xy-plane, subject to boundary conditions on ψ and ω, has been replaced by the problem of solving equations (15) and (16) for $y(x, \psi)$ and $\omega(x, \psi)$ in the rectangular $x\psi$-plane, described by $\{(x, \psi)|\psi_{\min} \leq \psi \leq \psi_{\max}; a \leq x \leq b\}$, subject to transformed boundary conditions on the variables y and ω. In both domains the equation of continuity is identically satisfied.

The advantage here is that in the von Mises coordinates one generates a rectangular domain in which the finite difference method can easily be applied when obtaining a numerical solution to the problem. It should be noted that the streamline pattern is obtained by plotting the values of $y(x, \psi)$ for all values of x and a given value of ψ.

In the transformed domain, boundary, inlet and exit conditions on the new dependent variable $y(x, \psi)$ are Dirichlet conditions: $y = f_1(x)$ on the lower boundary, and $y = f_2(x)$ on the upper boundary. At the inlet and exit, the values of y are obtained in terms of ψ at inlet and exit.

Vorticity on the solid boundary is obtained using equation (24). At the inlet and exit of the channel (in the computational domain), equation (24) is used to compute the vorticity from the inlet and exit velocity profile. Alternatively, at the exit one can use $\omega_x = 0$.

5 Transforming the Velocity-Vorticity Equations

The velocity-vorticity form of the governing equations (8), (9), and (10), is transformed into the von Mises variables using operators (13) and (14). The following equations are obtained:

$$L_2(\omega) = \omega\omega_\psi + Re\, u\omega_x \tag{25}$$

$$L_2(u) = \omega u_\psi - u\omega_\psi \tag{26}$$

$$L_2(v) = \omega_x + \omega v_\psi - v\omega_\psi \tag{27}$$

where

$$L_2 \equiv \partial_{xx} - 2v\partial_{x\psi} + q^2\partial_{\psi\psi} \tag{28}$$

and q^2 is the square of the speed, defined by equation (23), above.

It is required to solve the three equations (25), (26), and (27) for the three unknowns $\omega(x,\psi)$, $u(x,\psi)$ and $v(x,\psi)$. On the solid boundary, the zero-slip condition $u = v = 0$ is used. The vorticity is calculated numerically on the solid boundary using equation (24).

Equations (25-27), subject to boundary conditions on u, v, and ω, show an apparent independence from the shape of the solid boundary,$y(x,\psi_{\min})$ and $y(x,\psi_{\max})$. These only enter the problem formulation through the distribution of the horizontal computational grid lines and grid spacing $\Delta\,\psi$. For a given inlet profile, the spacing $\Delta\,\psi$ depends on the values of y at $x = a$. Thus, an infinite number of boundary shapes can be chosen such that the spacing and distribution of horizontal grid lines are the same for all configurations. This points to the existence of a class of configurations for which the inlet profile is the same, the grid spacing is the same, and the solution to the above equations is independent of the shape of the solid boundary.

This discussion furnishes the following observation.

Observation 3:

In the velocity-vorticity formulation, given by equations (25-27) in terms of the von Mises coordinates, there exists a class of problems for which the solution has an apparent independence from the shape of the boundary of the flow domain.

The preceding discussion resulted in the concept of universality of the von Mises transformation, coined by Hamdan and Ford [7]. The approach is summarized in the following steps:

1) Solve the velocity-vorticity equations (25-27) for u, v, and ω, subject to boundary conditions on these three variables. This solution (obtained only once) is valid for all boundary shapes if the same inlet velocity condition is used in all configurations.

2) For various boundary shapes, the flowfields are determined by solving the following Poisson equation for $y(x,\psi)$, subject to Dirichlet conditions on y:

$$y_{xx} + y_{\psi\psi} = \frac{uv_x - vu_x - u_\psi}{u^2}. \tag{29}$$

Alternatively, the following equation can be used to determine $y(x,\psi)$:

$$L_2(y) = \frac{\omega}{u} \tag{30}$$

where L_2 is given by equation (28).

Clearly, the shape of the boundary enters the formulation if one is required to determine the flow pattern.

The concept of a universal solution is attractive, as it can generate a large number of solutions by solving a single equation for $y(x,\psi)$, provided a one-time solution for velocity and vorticity has been obtained. However, this approach, as it stands, is incomplete and inaccurate as it attempts to solve a system of three equations (25-27) in the four unknowns, u, v, y and ω. Equations (29) and (30) are in fact identities that cannot be used to solve for the variable y. The resulting system is therefore under-determined. This implies that the computed velocity components do not necessarily satisfy the continuity equation. We summarize this in the following observation.

Observation 4:

1) Equations (29) and (30) are identities that cannot be used to solve for the variable $y(x,\psi)$. If the velocity components are defined in the von Mises variables by equations (18) and (19), then it follows that the right-hand-side of equation (29), or of equation (30), has the equivalent expression as given by the left-hand-side. This renders the velocity-vorticity formulation in von Mises variables incomplete.

2) Solution obtained in velocity-vorticity formulation does not satisfy the equation of continuity. The system of equations is under-determined (consisting of three equations in four unknowns). If the continuity is satisfied, then the boundary vorticity can be computed using equation (24). However, the absence of $y(x,\psi)$ from the equations hinders the satisfaction of continuity. For the continuity equation to be satisfied, we

state the following Theorem which can easily be established with the help of equations (18), (19), (20), and (23).

Theorem 1. *In the study of two-dimensional viscous fluid flow using the von Mises coordinate system, the continuity equation (20) is satisfied if and only if* $q^2 = \dfrac{1 + (y_x)^2}{(y_\psi)^2}$.

6 Conclusion

In this work we discussed the concept of universality of the von Mises transformation in an attempt to provide some corrections to a previous work that introduced the concept. We arrive at the following conclusions.

1) The von Mises transformation as a universal approach depends on casting the flow equations in velocity-vorticity form. Solution to these equations does not satisfy the equation of continuity. Furthermore, an equation for $y(x, \psi)$ is needed for completeness of the problem formulation.

2) Velocity-vorticity formulation must incorporate a condition on the satisfaction of equation of continuity. This condition is provided in Theorem 1. Clearly, the variable $y(x, \psi)$ must enter the formulation of the problem, otherwise the system of equations will be under-determined.

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