

# Extending Arcs: An Elementary Proof

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## Abstract

In a finite projective plane  $\pi$  we consider two configuration conditions involving arcs in  $\pi$  and show via combinatorial means that they are equivalent. When the conditions hold we are able to obtain embeddability results for arcs, all proofs being elementary. In particular, when  $\pi = PG(2, q)$  with  $q$  even we provide short proofs of some well known embeddability results.

## 1 Introduction

Let  $\pi$  be a projective plane of order  $n$  where  $n$  can be even or odd. An *arc* of size  $k$  or a *k-arc* in  $\pi$  is defined as a set of  $k$  points no three of which are collinear. If  $\mathcal{K}$  is a  $k$ -arc and  $P$  is a point with  $\mathcal{K} \cup \{P\}$  a  $(k + 1)$ -arc, we say that  $P$  is an *extending point* of  $\mathcal{K}$ . An arc is said to be *complete* if it possesses no extending points. A line  $\ell$  is said to be a *tangent* (resp. *secant*) of an arc  $\mathcal{K}$  if  $\ell$  meets exactly one point (resp. two points) of  $\mathcal{K}$ . In a plane of even (resp. odd) order an arc can have size at most  $n + 2$  (resp.  $n + 1$ ) in which case it is an *hyperoval* (resp. *oval*). For background on arcs in projective planes see [?] or [?]. Given a  $k$ -arc  $\mathcal{K}$  in  $\pi$  we define a parameter  $\delta$  associated with  $\mathcal{K}$  as follows:  $k + \delta = n + 2$ . We consider the following two conditions in  $\pi$ :

**Condition A:** Every arc of size  $k \geq \frac{n+4}{2}$  is contained in a unique complete arc.

**Condition B:** If  $\mathcal{K}$  is a complete  $k$ -arc, then no point of  $\pi$  lies on as many as  $\delta + 2$  tangents of  $\mathcal{K}$ .

For  $k \leq \frac{n}{2} + 3$  then Condition B is met trivially, so the condition is one on complete arcs of reasonable size. It is well known that the classical planes  $PG(2, q)$  where  $q$  is even

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meet Condition A ([?],[?]). Not all finite planes of even order satisfy Condition A; we mention a class of counterexamples due to Menichetti [?].

Our first result is that in  $\pi$  conditions A and B are equivalent. We call a plane meeting conditions A or B a *AB-plane*. In [?], Segre proved the following famous result:

**Theorem 1.** *In  $PG(2, q)$ ,  $q$  even, an arc of size  $k > q - \sqrt{q} + 1$  is contained in an unique hyperoval.*

In his proof, Segre used the very deep Hasse-Weil theorem. No elementary proof of Theorem ?? is known. Twenty years later, Thas [?] provided a proof of the following by employing elementary methods of algebraic and projective geometry.

**Theorem 2.** *In  $PG(2, q)$ ,  $q$  even, an arc of size  $k > q - \sqrt{q + \frac{1}{4}} + \frac{3}{2}$  is contained in an unique hyperoval.*

Based on this result Thas was able to give embedding results matching the bound of Segre in most cases. Here, we improve the bound in Theorem ?? in classical planes, and match the bound in the broader context of a general AB-plane. Our methods are elementary, combinatorial and self contained.

It is a long standing conjecture that complete  $q$ -arcs exist in all non-classical planes [?]. The weaker conjecture, that the bound in Theorem ?? holds only in classical planes, has also stood for twenty years [?]. Hence, it is of interest to investigate the existence of AB-planes that are non-classical. This question seems difficult and may be beyond present day techniques.

## 2 AB-Planes

**Theorem 3.** *In a finite projective plane, conditions A and B are equivalent.*

*Proof.* Let  $\pi$  be a projective plane of order  $n$  contain the complete  $k$ -arc  $\mathcal{K}$ . Suppose  $\pi$  is a AB-plane. Observe that each point of  $\mathcal{K}$  lies on  $k - 1$  secants and hence on exactly  $\delta$  tangents. Suppose by way of contradiction that  $P \notin \mathcal{K}$  lies on at least  $\delta + 2$  tangents. Then  $P$  is on say  $\alpha \leq \frac{k - (\delta + 2)}{2}$  secants. Form two new arcs  $\mathcal{K}'$  and  $\mathcal{K}''$  in the following manner. On each secant on  $P$  pick a point of  $\mathcal{K}$ , say  $P_1, P_2, \dots, P_\alpha$ . Let  $\mathcal{K}' = \mathcal{K} - \{P_1, P_2, \dots, P_\alpha\}$  and  $\mathcal{K}'' = \mathcal{K}' \cup \{P\}$ . Then both  $\mathcal{K}$  and  $\mathcal{K}''$  contain  $\mathcal{K}'$ , and

$$|\mathcal{K}'| = k - \alpha \geq k - \frac{k - (\delta + 2)}{2} = \frac{k + \delta + 2}{2} = \frac{n + 4}{2} \quad (1)$$

Since  $\pi$  is a AB-plane, the unique complete arc containing  $\mathcal{K}'$  must contain both  $\mathcal{K}$  and  $\mathcal{K}''$ . We conclude  $\{P\} \cup \mathcal{K}$  is an arc. This is a contradiction.

For the converse we assume  $\pi$  is not a AB-plane. So we have two complete arcs  $\mathcal{K}_1 \neq \mathcal{K}_2$  of size  $k_1$  and  $k_2$  respectively with  $|\mathcal{K}_1 \cap \mathcal{K}_2| \geq \frac{n}{2} + 2$ . Let  $\delta_2$  be defined by  $k_2 + \delta_2 = n + 2$  and choose  $P \in \mathcal{K}_1 - \mathcal{K}_2$ . Then  $P$  is on at least  $\frac{n}{2} + 2 \geq \delta_2 + 2$  tangents of  $\mathcal{K}_2$  and is not an extending point of  $\mathcal{K}_2$ .  $\square$

### 3 Arc Extending

**Lemma 1.** *Let  $\mathcal{K}$  be a  $k$ -arc in a plane  $\pi$  of order  $n$  where  $k > n - \sqrt{n + \frac{1}{4}} + \frac{3}{2}$ . Then on any tangent to  $\mathcal{K}$  there is a point  $P$  which lies on at least  $\delta + 1$  tangents. Furthermore, if  $n$  is even then  $P$  lies on at least  $\delta + 2$  tangents.*

*Proof.* Let  $\ell$  be a tangent of  $\mathcal{K}$  at say  $Q \in \mathcal{K}$ . There are exactly  $\binom{k-1}{2}$  secants intersecting  $\ell$  in points other than  $Q$ . As such there exists a point of  $\ell$  off of  $\mathcal{K}$  for which the maximum number of secants through that point is:

$$\binom{k-1}{2} \frac{1}{n} = \frac{(k-1)(k-2)}{2n} \quad (2)$$

We then have the number of tangents through some point of  $\ell$  off of  $\mathcal{K}$  is at least:

$$k - 2 \frac{(k-1)(k-2)}{2n} = (n - \delta + 2) - \frac{(n - \delta + 1)(n - \delta)}{n} = \delta + 1 - \frac{\delta(\delta - 1)}{n} \quad (3)$$

Thus, if  $n > \delta(\delta - 1)$  there is a point  $P$  of  $\ell$  off of  $\mathcal{K}$  which lies on at least  $\delta + 1$  tangents. A simple calculation with the restriction that  $\delta$  is non-negative then gives  $n > \delta(\delta - 1)$  iff  $\delta < \sqrt{n + \frac{1}{4}} + \frac{1}{2}$ . For the second part of the proof we observe that if  $n$  is even then  $k + \delta + 1 = n + 3$  is odd. So exactly one of  $k$  and  $\delta + 1$  is even. If  $k$  is even then  $P$  must be on an even number of tangents forcing  $P$  to be on at least  $\delta + 2$  tangents. If  $k$  is odd then  $P$  must be on an odd number of tangents forcing  $P$  to be on at least  $\delta + 2$  tangents.  $\square$

**Theorem 4.** *In an AB-plane of even order  $n$  a  $k$ -arc with  $k > n - \sqrt{n + \frac{1}{4}} + \frac{3}{2}$  is contained in a unique hyperoval.*

*Proof.* This follows immediately from Theorem ?? and Lemma ??.  $\square$

**Lemma 2.** *Let  $\mathcal{K}$  be an arc of size  $k > n - \sqrt{n + \frac{9}{4}} + \frac{3}{2}$  in a projective plane of order  $n = 2^t > 2$ . Then on any tangent to  $\mathcal{K}$  there is a point  $P$  which lies on at least  $\delta + 2$  tangents.*

*Proof.* Assume by way of contradiction that no point lies on as many as  $\delta + 2$  tangents. As in the proof of Lemma ?? we get

$$n \leq \delta(\delta - 1) \quad (4)$$

Since  $n = 2^t > 2$ , equality in (??) is not possible. Since both  $n$  and  $\delta(\delta - 1)$  are even, we get

$$n \leq \delta(\delta - 1) - 2$$

Simple calculations then give

$$n \leq \delta(\delta - 1) - 2 \Rightarrow k \leq n - \sqrt{n + \frac{9}{4}} + \frac{3}{2}$$

$\square$

**Theorem 5.** *In a AB-plane of order  $n = 2^t$  a  $k$ -arc with  $k > n - \sqrt{n + \frac{9}{4}} + \frac{3}{2}$  is contained in a hyperoval. If  $t > 1$  then the hyperoval is uniquely determined.*

*Proof.* In the case  $n = 2$  simple counting gives the result. The case  $t > 1$  follows immediately from Theorem ?? and Lemma ??.  $\square$

**Corollary 1.** *In  $PG(2, q)$ ,  $q$  even a  $k$ -arc with  $k > q - \sqrt{q + \frac{9}{4}} + \frac{3}{2}$  is contained in a unique hyperoval.*

**Remark 1.** *With reference to Theorem ?? we note the following inequality:*

$$\left( q - \sqrt{q + \frac{9}{4}} + \frac{3}{2} \right) - (q - \sqrt{q} + 1) = \frac{1}{2} - \frac{\frac{9}{4}}{\sqrt{q} + \sqrt{q + \frac{9}{4}}} < \frac{1}{2} \quad (5)$$

As in [?] we are able to improve our bounds in certain cases. For the sake of completeness we effectively reproduce the proofs in [?] for the following Lemmas.

**Lemma 3.** *If  $\pi$  is an AB-plane of even order  $n$  where  $n$  is a square, then every arc of size  $k > n - \sqrt{n} + 1$  is contained in a unique hyperoval.*

*Proof.* If  $n$  is a square, we have

$$k > n - \sqrt{n} + 1 \Rightarrow k > n - \sqrt{n} + \frac{3}{2} \Rightarrow k > n - \sqrt{n + \frac{9}{4}} + \frac{3}{2}.$$

$\square$

**Corollary 2.** *In  $PG(2, q)$ ,  $q = 2^{2t}$  every arc of size  $k > q - \sqrt{q} + 1$  is contained in a unique hyperoval.*

**Lemma 4.** *In a projective AB-plane  $\pi$  of even order  $n$ , every arc of odd size  $k$  satisfying  $k > n - \sqrt{n} + 1$  is contained in a unique hyperoval.*

*Proof.* Suppose that  $\mathcal{K}$  is a complete arc of odd size  $k < n + 2$ . By Theorems ?? and ?? no point of  $\pi - \mathcal{K}$  is on as many as  $\delta + 1$  tangents. Since  $k$  is odd, every point of  $\pi - \mathcal{K}$  must meet an odd number of tangent points. For each point  $P_i \notin \mathcal{K}$ ,  $1 \leq i \leq n^2 + n + 1 - k$ , denote by  $t_i$  the number of tangents on  $P_i$ .  $\mathcal{K}$  has  $k\delta$  tangents, each of which contains  $n$  points not from  $\mathcal{K}$ . So we have  $\sum_i t_i = nk\delta$ . By counting ordered triples  $(l, l', P)$  where  $l$  and  $l'$  are tangents through  $P \notin \mathcal{K}$  we obtain  $\sum_i t_i(t_i - 1) = k\delta(k\delta - \delta)$ . Since  $t_i$  is odd and less than  $\delta + 1$  we have  $\sum_i (t_i - 1)(t_i - \delta) \leq 0$  whence  $\sum_i t_i(t_i - 1) - \delta \sum_i t_i + \delta \sum_i 1 \leq 0$ . This gives  $k\delta(k\delta - \delta) - \delta^2 kn + \delta(n^2 + n + 1 - k) \leq 0$ . Substituting  $\delta = n + 2 - k$  and simplifying gives  $(k - 1)(k - 2 - n)(k^2 - 2k - 2kn + 1 + n^2 + n) \leq 0$  which in turn gives  $(k - 1)(n + 2 - k)(k - (n + \sqrt{n} + 1))(k - (n - \sqrt{n} + 1)) \geq 0$ . We conclude that  $k \leq n - \sqrt{n} + 1$ .  $\square$

**Corollary 3.** *In  $PG(2, q)$ ,  $q = 2^t$ , every arc of odd size  $k$  satisfying  $k > q - \sqrt{q} + 1$  is contained in a unique hyperoval.*

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