Bruck Nets and 2-Dimensional Codes

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Abstract

We introduce *net codes*, a class of $(n, 2, d)_q$ codes constructed using finite (Bruck) nets. We show linear codes to be precisely those net codes that may be constructed using Desarguesian nets. Consequently, we obtain some results on extending net codes, notably that linear codes of suitable length admit only linear extensions. We are also able to deduce that for certain values of q, nonlinear $(n, 2, d)_q$ codes are "short". Our proofs are combinatorial, using classical results on nets and planes that were introduced over thirty years ago.

1 Introduction and Definitions

In what follows, q need not be a prime power. An $(n, k, d)_q$-code C is a collection of q^k n-tuples (or codewords) over an alphabet $\mathcal{A}$ of size q such that the minimum (Hamming) distance between any two codewords of C is d (that is, there exist two codewords agreeing in $n - d$ coordinates and no two codewords agree in as many as $n - d + 1$.). In the special case that $\mathcal{A} = GF(q)$ (the finite field of order q) and C is a vector space of dimension k, we say that C is a *linear* $(n, k, d)_q$-code.

In searching for "long" codes a natural approach is to begin with a fixed code C and attempt to "lengthen" each codeword. One way of lengthening a code C is through the process of *repetition* whereby one or more coordinates of C are simply repeated. If C is an $(n, 2, d)_q$-code and C' is a code of length $n + 1$ obtained from C by repetition then C' will be either a $(n + 1, 2, d)_q$-code or a $(n + 1, 2, d + 1)_q$-code. As far as error correction is concerned, the latter case is more desirable.

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Definition 1.1. Let C be an $(n, k, d)_q$-code. A $(n+1, k, d+1)_q$-code C' is said to be an *extension* of C if upon deleting some fixed coordinate from each codeword of C' the code C is obtained. Equivalently, C is said to be *extendable* (to the code C'). A code is *maximal* if it admits no extensions.

The essence of a code extension is this: If C is an $(n, k, d)_q$ code then any two words in C agree in at most $t = n - d$ coordinates. In any extension C' of C, it is the case that any two codewords agree in at most t coordinates.

Let C be an $(n, 2, d)_q$-code. The code obtained by deleting some fixed coordinate from each codeword of C is called a *punctured code* of C.

Definition 1.2. Suppose C is a $(n, k, d)_q$-code. We define two types of operations on the codewords of C.

1. *positional permutation*: Fix two coordinate positions i and j. In each codeword of C interchange the i'th and j'th coordinate entries.

2. *symbol permutation*: Fix a coordinate position i and a permutation σ of the alphabet $\mathcal{A}$. Apply σ to the i'th coordinate entry of each codeword in C.

If a code C' can be obtained from a code C by a sequence of positional or symbol permutations then C' and C are said to be *equivalent*. If C' is linear then C is said to be *equivalent to linear*.

If two codes are equivalent then corresponding parameters such as minimum distance are equal, the codes are essentially identical. A code that is equivalent to linear need not be linear. For example, if we suitably permute the symbols in a given coordinate position of a linear code, the resulting code will not contain the zero vector and will therefore not be linear.

A *(Bruck) net* (see [4, 5]) $\mathcal{N}$ of order q and degree δ (or a (q, δ)-net) is a finite incidence structure of points and certain subsets of points called lines satisfying the following axioms.

(a) Any two points are incident with at most one line.

(b) Given a line ℓ and point P off ℓ, there is a unique line through P failing to meet ℓ. This line is said to be *parallel* to ℓ.

(c) There exist $\delta \geq 3$ parallel classes of lines (where two lines are in a common parallel class if and only if they are parallel), each consisting of q lines.

It follows that all lines of $\mathcal{N}$ contain exactly q points. Two lines from distinct parallel classes meet in a unique point. The total number of points in $\mathcal{N}$ is q^2 and the total number of lines is δq. A set T of q points in $\mathcal{N}$ is

called a *transversal* of $\mathcal{N}$ if no two points of T are incident.
To *extend* a net $\mathcal{N}$ is to append a parallel class of lines (thereby increasing the degree). It follows that $\mathcal{N}$ can be extended if and only if the points of $\mathcal{N}$ can be partitioned into transversals (forming the lines of a new parallel class). A net $\mathcal{N}_1$ is a *subnet* of the net $\mathcal{N}_2$ if $\mathcal{N}_1 = \mathcal{N}_2$ or if $\mathcal{N}_1$ can be successively extended to give $\mathcal{N}_2$. A (q, δ)-net is *maximal* if it not a subnet of any $(q, \delta+1)$-net. It can be shown that $\delta \leq q+1$ with equality if and only if $\mathcal{N}$ consists of the points and lines of an affine plane of order q. In particular, a $(q, q+1)$-net is maximal.

Let $\mathcal{N}$ be a net of order q. $\mathcal{N}$ is *embeddable* if $\mathcal{N}$ can be successively extended to an affine plane π of order q (called a *completion* of $\mathcal{N}$). $\mathcal{N}$ is a *Desarguesian net* if $\mathcal{N}$ is embeddable in $AG(2,q)$ (and therefore in $PG(2,q)$. The main result of Brucks 1963 paper is the following theorem.

Theorem 1.3. *(Bruck's Completion Theorem)*
Let $P(x) = \frac{1}{2}x^4 + x^3 + x^2 + \frac{3}{4}x$. *Then any* (q,δ)*-net* $\mathcal{N}$ *satisfying* $P(q-\delta) < q$ *has an unique completion.*

The key to the proof of this theorem is an ingenious "clique and claw" method first used by Bruck(1963) [5] and Bose(1963) [3]. The method has subsequently been used for many other embedding theorems, see for example Beutelspacher and Metsch [1, 2] for applications to linear spaces. The bound in Theorem 1.3 was improved by Metsch ([11] (1991)):

Theorem 1.4. *Let* $Q(x) = \frac{8}{3}x^3 - 6x^2 + \frac{11}{3}x + \frac{1}{3}$. *Then any* (q,δ)*-net* $\mathcal{N}$ *satisfying* $Q(q-\delta+1) < q$ *has an unique completion.*

In 1970, the results of Rédei [12] provided the following Theorem.

Theorem 1.5 (Rédei's Theorem)**.** *Let* $\pi = PG(2,q)$ *with a distinguished line* ℓ_∞. *Let* $\mathcal{S}$ *be a set of* q *points of* $\pi - \ell_\infty$ *and let* $\mathcal{A}$ *be a collection of* δ *points on* ℓ_∞ *with the following property. Any line through a point of* $\mathcal{A}$ *intersects* $\mathcal{S}$ *in at most one point. If*

$$\delta > \begin{cases} \frac{q-1}{2} & q \text{ is prime} \\ q - \sqrt{q} & \text{otherwise.} \end{cases}$$

then $\mathcal{S}$ *is a subset of a line of* π.

To the author's knowledge, though probably well known, the following has not appeared in the literature. In particular it gives Theorem 7 of [7]. We provide a short proof.

Corollary 1.6. *Let $\mathcal{N}$ be a Desarguesian (q,δ)-net where*

$$\delta > \begin{cases} \frac{q-1}{2} & q \text{ is prime, and} \\ q-\sqrt{q} & \text{otherwise.} \end{cases}$$

then $\mathcal{N}$ is is a subnet of precisely one maximal net, namely $\pi = AG(2,q)$.

Proof. Suppose $\mathcal{N}'$ is an extension of $\mathcal{N}$. Choose a line ℓ of $\mathcal{N}'$ that is not a line of $\mathcal{N}$. The points of ℓ form a transversal in $\mathcal{N}$. Consider $\mathcal{N}$ as embedded in $AG(2,q)$ and adjoin a line ℓ_∞ at infinity. Let $A = \{P_1, P_2, \dots, P_\delta\}$ be the set of points on ℓ_∞ incident with the lines of $\mathcal{N}$ (so $\mathcal{A}$ is the set of "slope points" of $\mathcal{N}$) . A transversal of $\mathcal{N}$ corresponds to a set of q points off ℓ_∞ such that any line through a point of $\mathcal{A}$ intersects the transversal in at most one point. From Theorem 1.5 it follows that ℓ (and therefore each line of $\mathcal{N}'$) is a line of $AG(2,q)$. The result follows. □

Remark 1.7. The second of the bounds in Corollary 1.6 is sharp. For example Hall planes of order q contain a Desarguesian net of order $q-\sqrt{q}$.

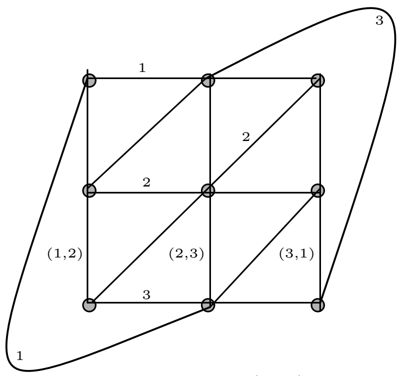
2 Net Codes

Example 1. In [13], Silverman shows (q,δ)-nets and $(\delta,2,\delta-1)_q$-codes to be combinatorially equivalent objects. Briefly, let $\mathcal{N}$ be a (q,δ)-net then each of the q^2 points of $\mathcal{N}$ gives a codeword as follows. Give an ordering $[\ell_1],[\ell_2],\dots,[\ell_\delta]$ to the δ parallel classes of $\mathcal{N}$. Within each parallel class, label the lines from $\{1,2,\dots,q\}$. Any fixed point P lies on an unique line from each parallel class. If P lies on say the line labelled α_i from parallel class $[\ell_i]$ for each i, $1 \le i \le \delta$ then we associate with P the codeword $(\alpha_1,\alpha_2,\dots,\alpha_\delta)$. Two points of $\mathcal{N}$ are on a common line if and only if the corresponding codewords share a common entry. The fact that two points of $\mathcal{N}$ lie on at most one line then gives the minimum distance $d = \delta - 1$.

We generalize Silverman's construction by allowing t-tuple labels on the lines of $\mathcal{N}$; a much broader class of codes can then be constructed. Specifically, to *t-label* a parallel class $[\ell] = \{l_1, l_2, \dots, l_q\}$ of $\mathcal{N}$ is to assign to each l_i a t-tuple $v_i = (a_{1i}, a_{2i}, \dots, a_{ti})$ over A such that for each j $1 \le j \le t$ there holds

$$\bigcup_{k=1}^{q} \{a_{jk}\} = A.$$

Example 2. Consider the $(3,3)$-net $\mathcal{N}$ pictured below where the vertical lines have been 2-labelled and the other lines 1-labelled. We construct a net code C of degree 3 and index 2 as follows. If $P \in \mathcal{N}$ is incident with the horizontal line with label h_1, the vertical line labelled (v_1,v_2) and diagonal line labelled d_1 then we deem that P shall correspond to the codeword $w(P) = (h_1, v_1, v_2, d_1)$. This yields the $(4,2,2)_3$-code C listed below (where the points $P_1, P_2, \ldots, P_9$ have been ordered from left to right, from top to bottom).



Point	Corresponding codeword
P_1	$(1,1,2,1)$
P_2	$(1,2,3,3)$
P_3	$(1,3,1,2)$
P_4	$(2,1,2,3)$
P_5	$(2,2,3,2)$
P_6	$(2,3,1,1)$
P_7	$(3,1,2,2)$
P_8	$(3,2,3,1)$
P_9	$(3,3,1,3)$

Figure 1: A $(3,3)-$net and a corresponding net-code.

Let C be an $(n,2,n-1)_q$-code constructed as in Example 1. If we 2-label a particular parallel class of $\mathcal{N}$ then the resulting code will be a $(n+1,2,n-1)$-code having C as a punctured code. By 2-labelling a second parallel class a $(n+2,2,n)_q$-code results. 2-labelling every parallel class gives a $(2n,2,2n-2)$-code.

We are ready to define an $(n,2,d)_q$-net code C: Beginning with a (q,δ)-net $\mathcal{N}$ we proceed as above, labelling parallel classes to construct a code. If a particular parallel class of $\mathcal{N}$ has been t-labelled then that parallel class is said to have *index* t. If the indices of the parallel classes of $\mathcal{N}$ are $t_1, t_2, \ldots, t_\delta$ respectively then it follows that $n = t_1 + t_2 + \cdots + t_\delta$. The maximum index among the parallel classes of $\mathcal{N}$ is the *index* of C. Necessarily the index of C is equal to $n-d$. C is of *uniform index* t if all parallel classes of $\mathcal{N}$ are of index t (in which case $n = \delta t$). An $(n,2,d)_q$ code that can be constructed in this way is a *net code* of *degree* δ *based on* the net $\mathcal{N}$. In particular, the code constructed in Example 1 is a net-code of degree δ and uniform index 1.

Let C be a $(n,2,d)_q$-net code of degree δ. Then two codewords of C agree in at most $t = n-d$ coordinates. From the Singleton bound it follows that an $(n,2,d)_q$-code (linear or not) satisfies $d \leq n-1$. Codes meeting this bound are called *Maximum Distance Separable* (MDS) (see [9]). The MDS codes of dimension 2 are precisely the net codes of index 1. Those $(n,2,d)_q$-codes satisfying $d = n-2$ are called Almost MDS codes (AMDS) (see [8]). All net codes of index 2 are therefore AMDS. The following two Lemma's are immediate consequences of the definition of a net code.

Lemma 2.1. *If C is an $(n, 2, d)_q$-code obtained from the net code C' by repetition, then C is a net code. Moreover, if C' is based on the net $\mathcal{N}$, then so is C.*

Lemma 2.2. *Let C be an $(n, 2, d)_q$-code obtained from the net code C' by puncturing. Then C is a net code. Moreover, if C' is based on the net $\mathcal{N}$, then C is based on a subnet of $\mathcal{N}$.*

Suppose C is an $(n, 2, d)_q$-net code based on the (q, δ)-net $\mathcal{N}$. Let the parallel classes $[\ell_1], [\ell_2], \ldots, [\ell_\delta]$ of $\mathcal{N}$ have indices $t_1, t_2, \ldots, t_\delta$ respectively. Assume with no loss of generality that $[\ell_1]$ determines the first t_1 coordinates of each codeword in C, $[\ell_2]$ the next t_2 coordinates, and so on. Consider an arbitrary codeword

$$w = (a_{11}, a_{12}, \ldots, a_{1t_1}, a_{21}, a_{22}, \ldots, a_{2t_2}, \ldots, a_{\delta 1}, a_{\delta 2}, \ldots, a_{\delta t_\delta})$$

If a codeword u in C has a coordinate, say a_{ij}, in common with w then u agrees with w in each a_{ik}, $k = 1, 2, \ldots, t_k$. Consequently, the punctured $(\delta, 2, \delta - 1)$-code C' obtained from C whereby

$$w \to (a_{11}, a_{21}, \ldots, a_{\delta 1})$$

is a (q, δ)-net (see Example 1), say $\mathcal{N}'$. Moreover, two points of $\mathcal{N}$ are incident if and only if the corresponding points of $\mathcal{N}'$ are incident. This gives us the following.

Lemma 2.3. *If C is a net code then the net upon which C is based is uniquely determined.*

Let C be a net code based on the net $\mathcal{N}$. Applying a symbol permutation to C simply amounts to permuting the labels assigned to each line of a particular parallel class of $\mathcal{N}$ whereas a positional permutation of C equates to a re-ordering of the parallel classes of $\mathcal{N}$. With this in mind, the following is immediate.

Theorem 2.4. *Let C_1 and C_2 be equivalent $(n, 2, d)_q$ codes. If C_1 is a net-code then so is C_2. Moreover, if C_1 and C_2 are based on the nets $\mathcal{N}_1$ and N_2 respectively then $\mathcal{N}_1$ and $\mathcal{N}_2$ are isomorphic (as nets).*

2.1 Linear Codes: Desarguesian Net-Codes

Let $GF(q) = \{a_1, a_1, \ldots, a_q\}$ and let C be a linear $(n, 2, d)_q$-code. Then C is equivalent to a code whose generator G has columns comprised entirely of vectors of the form $[\ a_i, 1\ ]^T$ or $[\ 1, 0\ ]^T$. In other words, C can be obtained by successive puncturing and/or successive repetition of the $(q, 2, q-1)$-code C' having a generator matrix

$$G' = \begin{bmatrix} a_1 & a_2 & \cdots & a_q & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \tag{2.1}$$

Let $\pi = PG(2, q)$ have homogeneous coordinates (x_1, x_2, x_3). Assume the line at infinity ℓ_∞ has equation $x_3 = 0$. Fix $n \leq q + 1$ and let $P_1, P_2, \ldots, P_n$ be points on ℓ_∞. Consider the (q, n)-net $\mathcal{N}$ in $\pi^* = \pi \setminus \{\ell_\infty\}$ having $P_1, P_2, \ldots, P_n$ as "slope points". Let C be a net code of index 1 based on $\mathcal{N}$. For each i, $1 \leq i \leq n$, let P_i be defined as the solution set of the following system of equations.

$$\begin{cases} x_3 = 0 \\ \alpha_i x_1 + \beta_i x_2 = 0, \quad \alpha_i, \beta_i \in GF(q) \end{cases} \tag{2.2}$$

Fix $P \in \mathcal{N}$ and for each i, $1 \leq i \leq n$ denote by $\ell_i(P)$ the unique line containing both P and P_i. Assume P corresponds to the codeword $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ in C. The code C will of course depend on the labelling of $\ell_i(P)$. To specify the labelling, let us proceed in the following way.

Assume for now that the point $Q = (0, 1, 0)$ is not among the P_i's, $1 \leq i \leq n$. Hence, in (2.2) we may assume $\beta_i = 1$, $1 \leq i \leq n$ and the equations become

$$\begin{cases} x_3 = 0 \\ \alpha_i x_1 + x_2 = 0 \end{cases} . \tag{2.3}$$

Let ℓ denote the (vertical) line with equation $x_1 = 0$. Then any point of π^* on ℓ has coordinates of the form $(0, \gamma, 1)$. $P = (a, b, 1) \in \pi^*$ is fixed. The line $\ell_i(P)$ meets ℓ in a point $(0, \gamma_i, 1)$ and γ_i gives the i^{th} coordinate entry of the codeword $(\gamma_1, \gamma_2, \ldots, \gamma_n)$ in C associated with the point P (see Figure 2).

Let us calculate γ_i. $\ell_i(P)$ has an equation of the form

$$\alpha_i x_1 + x_2 + \beta x_3 = 0 \tag{2.4}$$

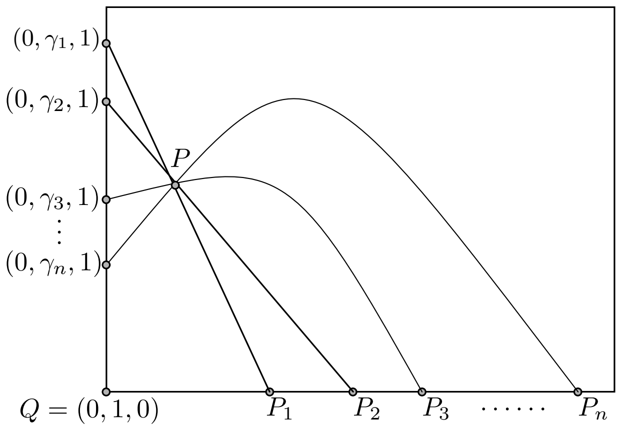


Figure 2: The Point/Codeword Correspondence.

As $\ell_i(P)$ contains $P=(a,b,1)$ then we have

$$\alpha_i a + b + \beta = 0$$

which gives

$$\beta = -\left(\alpha_i a + b\right) \tag{2.5}$$

The line given by (2.4) meets ℓ in the point $(0,\gamma_i,1)$ so that

$$\gamma_i + \beta = 0.$$

Therefore we have

$$\gamma_i = -\beta = \alpha_i a + b = \begin{bmatrix} a & b \end{bmatrix} \cdot \begin{bmatrix} \alpha_i \\ 1 \end{bmatrix}. \tag{2.6}$$

It follows that C is precisely the linear code with generator matrix

$$G = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Now, suppose the point $Q=(0,1,0)$ is among the P_i's, say $P_n=Q$. As above, $P=(a,b,1)$. For each i, $1 \le i \le n-1$, label $\ell_i(P)$ as above. The affine lines through P_n are the vertical lines. If the line ℓ' through P_n has equation

$$x_1 = \alpha$$

then we label ℓ' with α. It follows that the resulting net code C has a generator matrix of the form

$$G = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix}$$

It follows that the code C' with generator G' as in (2.1) is a Desarguesian net code of order q. As observed above, any linear code can be obtained through successive puncturing and/or repetition of C'. Therefore, Lemma's 2.1 and 2.2 and Theorem 2.4 give the following.

Theorem 2.5. *C is (equivalent to) a linear $(n, 2, d)_q$-code if and only if C is a Desarguesian net code of order q.*

For $q \leq 8$ it is known that the only projective planes of order q are Desarguesian. It is a long standing conjecture that all planes of prime order are in fact classical. By Theorem 1.4 any net of order q and degree $\delta > q + 1 - \epsilon$ where $\epsilon \approx \sqrt[3]{q}$ is embeddable in a projective plane. We get the following, which essentially says that for certain values of q, nonlinear codes are "short".

Corollary 2.6. *Let q be such that the only projective plane of order q is $PG(2, q)$ and let C be a $(n, 2, d)_q$-net code of degree δ. If $Q(q - \delta + 1) < q$ then C is equivalent to linear.*

3 Extending Net-Codes

In the majority of the literature only linear extensions of linear codes are considered. It is hypothesized that the best error correcting codes are in fact nonlinear. Thus, in considering code extension we take all possible extensions, linear or nonlinear into account. Recall that if C and C' are net codes and C' is an extension of C, then C and C' are of the same index.

Lemma 3.1. *Let C be a $(n, 2, d)_q$-net code. If C is not of uniform index, then C is not maximal.*

Proof. Let $\mathcal{N}$ be the net upon which C is based. If C is not of uniform index then there exists a parallel class, say $[\ell]$ of index $s < n - d$. Consider the s-tuples labelling the members of $[\ell]$. Repeat the final coordinate in each of these s-tuples. This gives an $(n+1, 2, d+1)_q$-net code C' extending C. □

Theorem 3.2. *Let C be a $(n, 2, d)_q$-net code of uniform index based on the net $\mathcal{N}$. Then C can be extended if and only if $\mathcal{N}$ can be extended. Moreover, any extension of C will be a net code based on some extension $\mathcal{N}'$ of $\mathcal{N}$.*

Proof. Let $A = \{a_1, a_2, \ldots, a_q\}$ be the alphabet over which C is defined. Denote by δ the degree of C and let $t = n - d$ be it's index. The if part of the theorem is clear. For the only if part let us suppose C can be extended to the $(n+1, 2, d+1)_q$-code C' (by appending an $n+1$'st coordinate position to each codeword of C). For each $a_i \in A$ denote by T_{a_i} the set of points of $\mathcal{N}$ corresponding to those codewords in C that when extended (in C') have a_i in the $n+1$'st coordinate position. We claim $|T_{a_i}| = q$ for each i, $1 \leq i \leq q$. Indeed, choose a parallel class $[\ell]$ of $\mathcal{N}$ and fix j, $1 \leq j \leq q$. Then for any line $\ell \in [\ell]$ we have $|\ell \cap T_{a_j}| \leq 1$ (else two codewords of C' agree in $t+1$ coordinates). The claim follows by the Pigeonhole Principle. The set $\{T_{a_1}, T_{a_2}, \ldots, T_{a_q}\}$ corresponds in the natural way to a partition of the points of $\mathcal{N}$ into transversals thereby extending $\mathcal{N}$ to say $\mathcal{N}'$. Moreover, C' is a net code of degree $\delta + 1$ based on $\mathcal{N}'$. □

Corollary 3.3. *If C is a $(n, 2, d)_q$-net code of uniform index and degree $q+1$ then C is maximal.*

If $\mathcal{N}$ is a maximal net then (Theorem 3.2) any net-code of uniform index based on $\mathcal{N}$ is maximal. Maximal nets of order q and degree $\delta < q+1$ exist. For example, in [6], Bruen shows that for prime p, there exist maximal nets of order p^2 and degree $\delta = p^2 - p$ or $p^2 - p - 1$. As such we get the following.

Lemma 3.4. *For $q = p^2$ where p is a prime, there exist maximal $(n, 2, d)_q$-net codes of degree $\delta = q^2 - \sqrt{q}$ or $\delta = q^2 - \sqrt{q} - 1$.*

By definition a Desarguesian net is embeddable in $PG(2, q)$. So a Desarguesian net code (in particular a linear code) can always be extended to a (necessarily maximal) Desarguesian net-code of degree $q+1$ and uniform index t. This gives the following Theorem, one corollary of which is Theorem 1 of [10].

Theorem 3.5. *Let C be (equivalent to) a linear $(n, 2, d)_q$ code of index t $(= n - d)$. Then C is maximal if and only if $n = t(q+1)$.*

A natural question arises: Is it possible for a linear code to admit a non-linear extension? As the following Theorem shows, for codes of reasonable length the answer is no.

Theorem 3.6. *Let C be (equivalent to) a linear $(n, 2, d)_q$-code, not necessarily of uniform index. Denote by $\mathcal{N}$ the Desarguesian net of degree δ upon which C is based. Let $[l_1]$, $[l_2]$, ..., $[l_m]$ be those parallel classes of $\mathcal{N}$ having index $t = n - d$. If*

$$m > \begin{cases} \frac{q-1}{2} & q \text{ is prime, or} \\ q - \sqrt{q} & \text{otherwise.} \end{cases}$$

then any extension of C is (equivalent to) linear and C extends uniquely to a maximal code.

Proof. Denote by $\mathcal{N}'$ the (q, m)-subnet of $\mathcal{N}$ having parallel classes $[l_1]$, $[l_2]$, …, $[l_m]$. It follows (as in the proof of Theorem 3.2) that any extension C arises via an extension of $\mathcal{N}'$. By Corollary 1.6, any extension of $\mathcal{N}'$ is a Desarguesian net. Consequently, all extensions of C are equivalent to linear and C extends uniquely to a maximal code. □

Corollary 3.7. *If C is (equivalent to) a linear $(n, 2, d)_q$-code of uniform index and degree δ where*

$$\delta > \begin{cases} \frac{q-1}{2} & q \text{ is prime, or} \\ q - \sqrt{q} & \text{otherwise.} \end{cases}$$

Then any extension of C is (equivalent to) linear and C extends uniquely to a maximal code.

Corollary 3.8. *If C is (equivalent to) a linear $(n, 2, d)_q$-code of index t $(= n - d)$ and*

$$n > \begin{cases} (q+1)(t-1) + \frac{q-1}{2} & q \text{ is prime, or} \\ (q+1)(t-1) + q - \sqrt{q} & \text{otherwise.} \end{cases}$$

Then any extension of C is (equivalent to) linear, and hence C extends uniquely to a maximal code.

Proof. Let $\mathcal{N}$ be the Desarguesian net upon which C is based. The degree δ of $\mathcal{N}$ is at most $q + 1$ and each parallel class of $\mathcal{N}$ is of index at most t. C then satisfies the hypotheses of Theorem 3.6 and the result follows. □

Corollary 3.9. *If C is (equivalent to) a linear $(n, 2, d)_q$-code of index 1 (i.e. a 2-dimensional MDS code) and*

$$n > \begin{cases} \frac{q-1}{2} & q \text{ is prime, or} \\ q - \sqrt{q} & \text{otherwise.} \end{cases}$$

Then any extension of C is (equivalent to) linear, and hence C extends uniquely to a maximal code.

Corollary 3.10. *If C is (equivalent to) a linear $(n, 2, d)_q$-code of index 2 (i.e. a 2-dimensional AMDS code) and*

$$n > \begin{cases} \frac{3q+1}{2} & q \text{ is prime, or} \\ 2q - \sqrt{q} + 1 & \text{otherwise.} \end{cases}$$

Then any extension of C is (equivalent to) linear, and hence C extends uniquely to a maximal code.

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